Counterpoint Worlds and Morphisms

A Graph-Theoretical Approach and its Implementation on the Rubato Composer Software

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Abstract

The mathematical modeling of classical counterpoint in part VII of the *Topos of music* [MGM02] complements the classical Fux system [Fux25] by five other "exotic" systems of rules for composition. They all derive from the same algebraic model, but only the traditional Fuxian system has been used by musicians, leaving five unexplored counterpoint worlds for the composition of new music. In this dissertation, we present and discuss concepts, algorithms, and the implementation in software modules of the Rubato Composer system for the possible morphing transitions between these systems. This enables the creation of exotic contrapuntal compositions by structural transport.

To do so, we represent authorisations and interdictions within different counterpoint systems by directed graphs. Morphing contrapuntal compositions from one world into contrapuntal compositions compliant with the rules of another world then reduces to the construction of a set of distinguished digraph morphisms. This shows that these worlds do not only coexist independently: a procedure is given to morph music from one world into an other, thus connecting them into a network of contrapuntal travels.

Using the Rubato implementation of [Mil06], composers can create and manipulate counterpoints by exploring a complete catalogue of possible transformations.

Keywords

Counterpoint, consonance, dissonance, symmetry, graph theory, category theory, computer assisted composition.

Zusammenfassung

Das im Kapitel VII vom *Topos of Music* [MGM02] vorgestellte mathematische Modell für den Kontrapunkt der ersten Gattung bereichert die klassischen Fuxschen Regeln um fünf weitere "exotische" Systeme von Kompositonsreglen. Alle lassen sich vom selben algebraischen Modell herleiten, aber nur das traditionelle Fuxsche System wurde bisher von Musikern angewendet. Es bleiben also fünf musikalische Welten für die Komposition neuer Musik offen. In dieser Arbeit werden Konzepte und Algorithmen diskutiert, die zu einer Implementation in der Rubato Software geführt haben, der es jetzt gelingt, Musikstücke zwischen den verschiedenen Welten zu transformieren, d.h. exotische Kontrapunkte durch Übertragung der Struktur zu erzeugen. Um dies zu erreichen, werden erlaubte und verbotene Übergänge zwischen den Intervallen eines Kontrapunktes durch gerichtete Graphen dargestellt. Die Transformation eines Kontrapunkts von einer Welt in die andere wird dann auf die Konstruktion eines Graphenhomomorphismus reduziert. Die Analyse dieser Prozedur demonstriert zugleich, dass die Kontrapunktwelten nicht isoliert sind. Sie bilden sogar ein ganzes Netzwerk von Kontrapunktischen Reisen. Mittels der Rubato Software können nicht nur exotische Kontrapunkte erzeugt werden, sondern auch die ganze Skala möglicher Transformationen erforscht.

Schlagwörter

Kontrapunkt, Konsonanz, Dissonanz, Symmetrie, Graphentheorie, Kategorientheorie, Komputergestützte Komposition.

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Questions raised in this field at the convergence of mathematics, music and philosophy are just as stimulating now, if not more so, than at any time over the past two thousands years, and I really enjoyed the many discussions we had at the MultiMedia Lab in Zurich, including those with Florian Thalmann who was there at that time for his Master Thesis.

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Table 1: Logical and set-theoretical symbols

Symbol	Meaning
$\top$	True
$\perp$	False
$\wedge$	And
$\vee$	Or
$\uplus$	Disjoint union
$\setminus$	Difference

Table 2: Symbols for Categories

Symbol	Meaning
$\mathfrak{C}_0$	Objects of a category $\mathfrak{C}$
$\mathfrak{C}_1$	Morphisms of a category $\mathfrak{C}$
$\mathrm{Dom}\,(f)$	Domain of a morphism f
$\mathrm{Cod}\,(f)$	Codomain of a morphism f
$\cong$	Isomorphic
$Nat\,(\mathcal{F}, \mathcal{G})$	Natural transforms from functor $\mathcal{F}$ to $\mathcal{G}$
$\mathfrak{Set}$	Category of sets
$\mathfrak{Grp}$	Category of groups
$\mathfrak{Gph}$	Category of graphs
$\mathfrak{DGph}$	Category of digraphs
$\mathfrak{SDGph}$	Category of strict digraphs
$\mathfrak{Cir}$	Circle category
$\mathfrak{Tor}$	Torus Category
$\mathfrak{Mod}$	Category of modules
$\mathfrak{Path}$	Category of Paths
$\mathfrak{Ctp}$	Counterpoint category

Table 3: Mathematical symbols

Symbol	Meaning
ϕ	Euler's totient function
ZD	Zero divisors
$\langle m, n\rangle$	Greatest common divisor of m and n
$\mathbb{N}$	Natural numbers (zero included)

Table 4: Chromatic cicrle

Symbol	Meaning
n	Chromatic gamut size (number of notes in an octave)
$\mathcal{C}_n$	Chromatic circle with n notes
$[k]_n$	Pitch or interval class k modulo n
${\mathcal{C}_n}^{\times}$	Units of the chromatic circle
ϕ	Circle symmetry
$\mathbb{Z}_n$	Cyclic group of order n
$\mathbb{D}_n$	Dihedral group of order $2n$
$\mathbb{A}_n$	Affine ismorphisms group
$\mathbb{I}_n$	Torus isometry group
$[\Delta]_{\mathbb{I}_n}$	Strong dichotomy class
$Sym\,(n)$	Symmetric group (permutations of n elements)

Table 5: Symbols used for denoting elements used in the context of counterpoint

Symbol	Meaning
Δ	Strong dichotomy
$[\Delta]_{\mathbb{I}_n}$	Strong dichotomy class
k	Consonance
K	Consonance set
d	Dissonance
D	Dissonance set
x	Cantus firmus pitch
z	Discantus pitch
y	Interval
$\mathcal{C}_n[\varepsilon]$	Dual space
ζ	Contrapuntal interval
ξ	Contrapuntal consonance
η	Contrapuntal dissonance
$\hat{\cdot}$	Contrapuntal orientation
h	Contrapuntal symmetry
p^x_Δ	Local autocomplementary function
$p^\bullet_\Delta$	Global autocomplementary function
p	Counterpoint
s	Step
CW	Counterpoint world
κ	Consonance indicator function
σ	Step indicator function
$\mathcal{R}$	Relevant contrapuntal consonances
ψ	Counterpoint world morphism

Table 6: Symbols used for denoting elements used in the context of graph theory

Symbol	Meaning
D	Directed graph
$D[S]$	Subgraph induced by S
T	Tree
K_n	Complete graph of order n
V	Vertex set
A	Arrow set
$N_v\left(G\right)$	Neighbourhood
$N_v^{(0)}\left(D\right)$	Predecessor set
$N_v^{(1)}\left(D\right)$	Successor set
n_1	Successor count
n_0	Predecessor count
$a^{(1)}$	Out-degree
$a^{(0)}$	In-degree
ϕ	Graph homomorphism
D	Strict digraph
ϕ	Strict digraph morphism
$\mathcal{Q}$	Quotient graph functor
$\mathcal{P}$	Super quotient graph functor (parent)
$\mathcal{N}$	Null quotient graph functor
$\mathcal{W}$	Weak quotient graph functor
$\mathcal{S}$	Strong quotient graph functor
$\mathcal{H}$	Homogeneous quotient graph functor
$\mathcal{F}$	Full quotient graph functor

Chapter 1

Introduction

The mathematical model of counterpoint developed in Guerino Mazzola's *Geometrie der Töne* [Maz90] reproduces the corpus of rules gathered by Johann Jospeh Fux in his classical treatise *Gradus ad Parnassum* [Fux25]. In addition to the rules of counterpoint empirically developed by Western musicians over centuries, five further systems of musical composition can be mathematically synthesised in the twelve semitones context. These *exotic* counterpoint worlds are not known to have been used by musicians so far, as mentioned in Chap. 15 and 16 of *La Vérité du Beau dans la musique* [Maz07], or at the end of Sec. 31 in *The Topos of Music* [MGM02].

The present work is about exploring new systems of musical composition, especially the transformation of counterpoints between the Fuxian and exotic worlds. Before explaining our intentions further in Sec. 1.2, a brief summary of both traditional and mathematical counterpoint is given in Sec. 1.1. A bibliography containing resources in music theory (the field of application) and graph theory (the methods used in this dissertation) is found in Sec. 1.3. Sec. 1.8 describes how the next chapters are organised.

1.1 Counterpoint

Polyphony has played a key role in Western music for over a thousand years. Voice leading was a main concern to composers during the Renaissance, and the principles they developed over centuries were collected in what is now called *counterpoint*, a set of rules solving two problems caused by polyphonic music: the interdependencies of voices and the alternation between *consonances* and *dissonances*. This term commonly serves to designate two different things: the method and the result, that is, the theoretical framework, or set of rules, and the music composed in accordance with it.

The name comes from the latin *punctum contra punctum*: point against point, and defines how to conduct several parallel voices. Even if each voice evolves independently, global coherence must be maintained. In this context, *harmony* is thought of as unity in diversity and concerns only the intervals between voices. The concept will gain its modern signification and apply to chords much later, between the 17th and 18th centuries.

This evolution expresses the difference between melody and harmony, horizontal versus vertical thinking, heterophony (all voices being independent and equal) or homophony (simultaneous notes merging into a single homogeneous chord). It is of course hard to apply one of these extreme principles exclusively, and music very often contains a mixture of both. Johann Sebastian Bach is one of the most celebrated musicians for having used counterpoint techniques integrating harmonic content.

History

The evolution of counterpoint starts in the Middle-Ages and stretches over more than six centuries, up to the Renaissance.

The first examples of polyphony in Western music date from the 9th century. It began as an enrichment of monodic Gregorian chant, and the earliest attempts to add a second voice were simple transpositions a fifth apart of a copy of the original voice, called the *Cantus firmus*. The motion was parallel and the rhythm identical, yielding and enrichment of timbre, not of the melodies.

Polyphonic works of the Middle-Ages are called *organum*. The origin of this term is not clear and specialists argue different etymological hypotheses. One controversial explanation pretends that it could come from the Latin word for a musical instrument, apparently because, unlike the human voice, it is possible for an instrument to produce different simultaneous sounds.[1] In Greek it means measurement tool and the word could be connected with the idea of harmony. The contrary motion of voices appears during this period, but still uses exclusively the traditional Pythagorean consonances: fourth, fifth, and octave. The second voice then becomes independent and gets its own name: the *discantus*. The repertoires of the cathedrals of Notre-Dame in France and Winchester in England constitute the greatest corpus left. These centres were active until the 12th century; their style was later replaced by the *motets*.

An increase in rhythmical complexity, for example doubling the speed of the discantus, could be achieved in the 14th century thanks to the new notational and compositional techniques introduced by the *Ars Nova*. Counterpoint eventually reaches its culmination with the work of Giovanni Pierluigi da Palestrina (1525–1594).

One of the problems not solved by this theory is unity within a piece of music, a reason which may explain why composers moved further towards tonality, explains Nicolas Meeùs in [Mee07], page 120. As a reaction against the tonal system, horizontal thinking reappears in the 20th century: Claude Debussy, Bela Bartók, Igor Stravinsky, Paul Hindemith, or Olivier Messiaen among others saw there an alternative to the vertical conception of music, and Miles Davis's *So What* inaugurated *modal jazz* in 1958.

Styles and Voices

It was usual to start with a given voice, the *Cantus firmus*, and add new voices as ornaments one after the other, most often up to four. Two famous examples exist where this technique was pursued further: Johannes Ockeghem's *Deo Gratias* (15th century) is a 36-part canon, and Thomas Tallis 40-voice piece *Spem in alium* (1573).

[1] Diphonic singing as found in Mongolia, Tibet, or South Africa, being an exception.

The Baroque era of the 17th century saw the *ancient style*, or *stile antiquo*, designating the old-fashioned writing of church music *à la Palestrina*, being distinguished from the *stile moderno* where new techniques were used notably for operas and sonatas.

The *strict style*, or *stile severo* is a term which gained acceptance in the 14th century. It means that the number of voices does not vary: they all start together and play together until the end. No voice appears nor disappears during the piece.

Species

Counterpoints are classified into different *species*, defined by the number of voices and rhythmic structure. Fux used them to segment and organise his treatise in teaching units of increasing difficulty [Fux25]. Typical exercises consist in adding new voices to a given one.

Progressively augmenting the number of voices, and adding rhythmic complexity, Fux covers counterpoints from the first to the fifth species, as shown in Tab. 1.1. Species are conceived as building a (pedagogical) stair which guides the student to the mastery of counterpoint techniques, hence the title *Gradus ad Parnassum*.

He was not the first to segment his treatise in this way (see the list of his predecessors in Sec. 1.5) and other systems including even more species were proposed after him. But his pedagogical taxonomy of exercises was the first to be as systematic and serves even nowadays as a current classification.

Table 1.1: The five different species of counterpoint, as presented in Chap. II of *Der neue Gradus* [Tit59]. The ratio $n : m$ designates the number of notes n played by the voices to add, versus the number of notes m played by the given cantus firmus. Species are ordered by increasing difficulty, and the last one encompasses the previous ones by freely combining all the rules.

Species	Voices	Discantus Rhythm	Intervals
First	2	*Nota contra notam*, whole notes $(1 : 1)$	Cons.
Second	2	Half $(2 : 1)$ or third $(3 : 1)$ notes	Diss. on weak beats
Third	2	Quarter notes $(4 : 1)$	
Fourth	4	Syncopations, ligatures	Diss. on strong beats
Fifth	≤ 4	*Contrapunctus floridus*	

Musically Relevant Parameters

The complexity resulting from the stacking of voices focuses the attention on pitch, leaving little or no more room for any other musical parameter. The rhythm has to be kept simple: this kind of polyphonic music is sung in churches where acoustics impose severe limitations on tempo and rhythmic complexity. Complex or varying timbres are avoided: composers favoured instruments continuously producing a rather pure sound, such as the human voice, brass, and bowed string instruments.

Rules

Rules governing counterpoint enter broadly into two categories.

1. A set of authorisations and interdictions dictating which intervals are allowed to separate the two voices. This is the easy part, it boils down to remembering an immutable list of consonances and dissonances.

2. The set of rules dictating how consonances are allowed to follow each other through the piece is numerous. These rules depend on context and constitute the hardest part to learn.

There are also some rules that shape the melody of a voice and are determined by singing techniques (e.g. prohibition of great intervals that are hard to sing, restriction to a particular range of the human voice) but we will ignore them here. Such considerations could be integrated on a practical level without affecting the theory.

None of the rules has an acoustic origin (such as the resonance of the harmonics). The dichotomy, or distinction, between consonances and dissonances constitutes the heart of the system, as noted by Meeùs, especially when it comes to movements involving joint motion and resolution of dissonances [Mee07]:

> Il faut insister sur le fait qu'aucune des règles du contrepoint n'est dictée par la nature du son lui-même, aucune n'est d'origine acoustique. La seule donnée objective, physique, sur laquelle elles reposent, c'est la différentiation entre consonance et dissonance.[2]

The Mathematical Model

Counterpoints handled by Mazzola's theory concern pieces for two voices, in the strict style and first species. This means that the cantus firmus and the discantus always play note against note, forming a sequence of intervals, see Fig. 1.1 for an example of such a counterpoint.

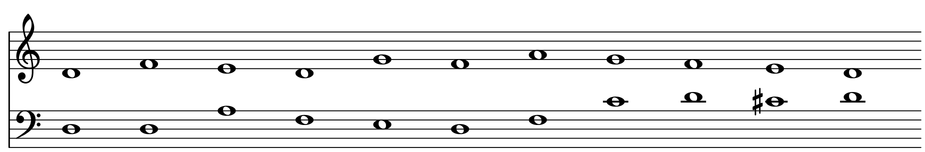

Figure 1.1: An example of a dorian counterpoint taken from Ernst Tittel [Tit59], example number 66b, page 12. Cantus firmus on the upper staff, discantus below.

The model is based on an algorithm which generates rules, completing the authorisations for intervals (the consonances, given at the input) with the authorisations for their succession (the allowed steps produced at the output).

[2]We have to insist on the fact that no rule of counterpoint is dictated by the nature of sound itself. None originates in acoustics. The only objective, physical element they rely on is the differentiation between consonance and dissonance.

The procedure is not restricted to the Fuxian rules. Many different sets of consonances form a valid input,[3] each one yielding its own system of rules on composition, which we will define as a *counterpoint world* in Chap. 3.

1.2 Problem Statement

The aim of this dissertation is to investigate the existence and structure of *morphisms*, or transformations, between counterpoint worlds. The main question is: is it possible at all to build a morphism linking counterpoint worlds? And how could we then transform a piece of music from one world to another? For example, can we choose a two voice counterpoint from Bach living in our well-known Fuxian world, and change its pitches to obtain a new piece of music while preserving its contrapuntal structure?

We could restate the problem using concepts from category theory. We know that there exist six types of counterpoint worlds, which form the objects of a category. But the existence of morphisms between the worlds has not been investigated so far. We thus want to complete the category of counterpoint worlds by providing it with morphisms.

Once demonstrated the existence of mappings between different worlds, we want to find a way to construct them explicitly. In order to do that, new theoretical concepts have to be added to the existing algebraical model of counterpoint, and algorithms have to be formulated so that an implementation in Rubato Composer, Gérard Milmeister's functorial music theory software [Mil06], becomes possible.

1.3 Related Works

The main sources and theoretical foundations of this dissertation are found in Mazzola's three books treating the subject. The modelling of Fuxian counterpoint was first exposed in *Geometrie der Töne* [Maz90]. Part VII of *The Topos of Music* [MGM02] contains its generalisation to new systems of composition. It is the most complete text, formulated in terms of *forms* and *denotators*, the universal data format of functorial music theory, inspired by category theory, see Chap. 6. This formalism was later implemented in the Rubato Composer software, see Milmeister's complete documentation [Mil06] for more details. Because we will continuously refer to this text through this thesis, the reader will encounter the abbreviation *ToM* to denote it. *La Vérité du Beau dans la musique* [Maz07], leaves most mathematics in the background and discusses philosophical implications. The concepts and methods contained in this dissertation were presented for the first time at the ICMC 2007 in Copenhagen [JM07].

On the one hand, the differentiation between *consonances* and *dissonances* belongs to the core of mathematical counterpoint theory, as will be shown in Sec. 2.1 and 3.1. Indeed, the classification into more or less consonant intervals has been a key concern among Western music theorists since Antiquity and caused passionate debates. Sec. 1.4 proposes a brief survey of the numerous of attempts made to build a scientific theory of consonances.

[3] Sec. 2.1.3 tells exactly which ones qualify.

On the other hand, classical treatises on counterpoint have been written exclusively by musicians, Sec. 1.5 lists some of them. Possible computer science applications to counterpoint, including essentially algorithmic composition and teaching, are enumerated in Sec. 1.6. Finally, Sec. 1.7 cites documents on graph theory, especially morphisms and drawings, which are the main methodological issues addressed in this dissertation.

1.4 A Brief History of Consonances and Dissonances

Why do we enjoy certain intervals more than others? Can we classify them, can we measure their *sweetness*? Which and how many consonances are there anyway? Those questions soon arise when one looks into consonances and dissonances, and music theorists have imagined many different, often contradictory answers to them over the past two thousand and five hundred years.[4]

This field is not only of primary interest for musicologists, but to mathematicians and physicists too. The reader can find many texts concerning the relation between the sciences and music theory: Volume Two of Barker for Greek Antiquity [Bar04], Palisca for the Renaissance [Pal85], Cohen for the Classical Age [Coh84], as well as two articles from Dostrovsky [Dos75] and Truesdell [Tru60]. A summary of these *objective* models covering the period from the Greeks to the 20th century can be found in Bailhache's book [Bai01]. It served as a great source of inspiration for this section, which is more or less a summary of this book, while Laurent Fichet focuses on the last two centuries [Fic95]. More recently, Muzzulini published an extensive coverage of theories of the last four centuries, in german [Muz06]. Zanoncelli's article on recurrent subjects in music theory [Zan06] gives a survey up to the Renaissance, and Dalhaus should also be mentioned [DM84] for a general treatment of the subject.

The aim of this section is not to provide a comprehensive discussion about *consonance* and *dissonance*, a very broad topic, but to remember a few key debates in order to illustrate the complexity of a theoretical problem that has never been solved. This overview of scientific formulas will allow us to understand better into which tradition of thought the *strong dichotomies* of mathematical counterpoint theory defined in Sec. 2.1.3 fit, and what makes their originality.

The authors who tried to settle the concept of *consonance* on a firm scientific basis, have been facing three issues:

1. The existence and definition of consonances

2. The total amount of consonances

3. The definition of dissonances, as excluded from the consonances

The many attempts to answer these questions broadly fall into two categories: whether reason is favoured, or perception and emotion. We will concentrate on the so-called rational ones, the scientific theories. Since the study of music involves so many different

[4] See Tab. 1.2 at the end of this section.

fields,[5] some models are based on purely numerical considerations, others concentrate on the physical nature of sound, and a third school of thought handles the physiological and psycho-acoustical dimension.

It is also possible to completely ignore physical realities and to prefer human perception and emotion. This is what most musicians do, or believe they do, and feel perfectly happy with it. The debate between scientists and artists always boils down to a conflict between *naturalistic* ("objective") and *culturalistic* ("subjective") approaches, and most theories lie somewhere inbetween, sometimes trying to accommodate two extremes: pure number mysticism on one side and Romanticism due to an idealistic view of artistic inspiration on the other side, as discussed in [Fic95]. This dichotomy has influenced most of Western thinking over the past two thousand years.

The next sections contain a survey of different types of explanatory models, as well as a summary of critics. The case of the fourth will show how versatile musical thinking and practice can be, and how compositional techniques can also influence the popularity of an interval.

1.4.1 Numerical Models

Mathematics and especially geometry have been present in Western music theory since the Greeks, and progressively lost their importance in favour of acoustics. This situation will change again during the 20th century, where atonal combinatorics of *Set theory*, as well as computer science, will reintroduce mathematical tools, widening a field of application traditionally confined to consonances, scales, and tuning.

Historically, quantitative methods have been of cardinal importance for describing pitches, the principal parameter of interest in Western music.

> Without number and measurement, any art is at the mercy of musical guesswork and of an empirical reliance of the senses.[6]

The number—ancient Greeks only knew finite, positive integers—appears as a symbol and the key to the comprehension of phenomena. In its essence, the basic idea is the following: the simpler the ratio of string lengths, the greater the harmony of an interval. There are of course additional subtleties, but Gottfried Wilhelm Leibniz (1646–1716) shares this point of view centuries later:

> Musica est exercitium arithmeticae occultum nescientis se numerare animi.[7]

The following paragraphs will show that this is more easily said than done.

Antiquity

In Western tradition, the Pythagoreans, in the 5th century BC, are credited with the oldest connection between mathematics and music. A famous legend tells how Pythago-

[5] See for example the propositions for a systematic musicology, beginning at page 11 of Guido Adler's seminal article in [Adl85].

[6] Plato, *Philebus*, §55e–56a [PDt93].

[7] Music is an occult practice of arithmetics in which the soul does not notice it is counting, cited from a letter to Goldbach dates 17th April 1712.

ras, after having heard the harmonious noise made by working blacksmiths, started work on his music theory, see fragment 9 of Xenocrates (c. 396–314). The Pythagorean school defines five consonances, expressed by simple ratios:

$$\begin{array}{ll} 2:1 & \text{octave} \\ 3:2 & \text{fifth} \\ 3:1 & \text{octave plus fifth} \\ 4:3 & \text{fourth} \\ 4:1 & \text{double octave} \end{array} \tag{1.1}$$

where $n:m$ defines the ratio linking two notes in an interval. The values available for the numerator and the denominator have to be positive integers, and are not allowed to exceed 4:

$$m, n \in \{1, 2, 3, 4\} \tag{1.2}$$

which is the sequence of numbers of points in the *tetraktys*, a Pythagorean symbol shown in Fig. 1.2.

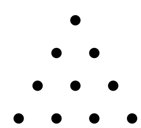

Figure 1.2: The Tetraktys, a Pythagorean triangular symbol for the Cosmos, the four elements, along with other geometrical aspects of space. It also served as a base for their musical system.

Intervals are classified according to the nature of the ratio. The *genus multiplex*

$$(k \cdot n) : n \quad n, k \in \mathbb{N} \tag{1.3}$$

provides the best ratios: $2:1$, $3:1$, $4:1$. Then follows the *genus superparticularis*

$$(n+1) : n \quad n \in \mathbb{N} \tag{1.4}$$

containing the fifth $(3:2)$ and the fourth $(4:3)$. The rest of the intervals belong to the *genus superpartiens*

$$m : n \quad m, n \in \mathbb{N} : m > n+1 \tag{1.5}$$

The genus superparticularis is linked to the even integers by means of the *gnomon*, a graphical representation of numbers, see Fig. 1.3.

Pythagoras did not leave any written material. His thoughts (or those of his colleagues and pupils) were transmitted by indirect citations. Plato (429–347) is the one who popularised Pythagorean ideas for later theorists. On one hand he criticises their obsession for numerical aspects of consonances in the *Republic*, 531b–c [PMt02], but on the other hand his dichotomy between ideal and real worlds relates to Pythagoras by the mean of proportions, an expression of harmony. The same mathematical model serves for the construction of soul and world. Human music imitates the harmony of

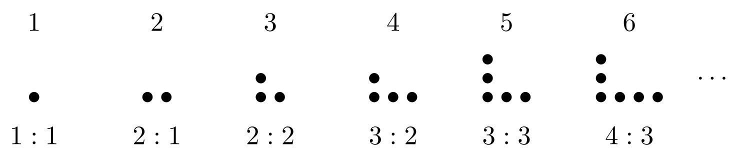

Figure 1.3: The gnomon, a graphical representation of integer numbers. Pebbles are aligned along two perpendicular axes. Odd numbers always form two adjacent sides of a square, defining a unique ratio $n : n$. Even numbers, on the other hand, define an infinity of rectangular shapes described by superparticular ratios $(n + 1) : n$.

spheres, the ideal expression of order and concordance which is realised by the planets. They revolve on rings or spheres, and their velocity produces musical tones, as told in the myth of Er in the *Timaeus*, 617b, 88 and 80b [PRt02]. This should not be understood as real music (since our birth we mortals hear it all the time without noticing it), but as the foundation of the Greek science of harmonics. Tuning a monochord according to the Pythagorean system is quite easy. And together with these deep philosophical implications, the prevalence of numbers in music theory will be established for a long time.

In his famous *Elements of Geometry*, Euclid (around the 4th and 3rd centuries BC) could establish geometry on a firm rational ground by constructing it systematically, on a basis of axioms and theorems. The *Division of the Canon* is an attempt to reproduce the same rigourous and successful approach, this time applied to music. He demonstrates that an interval is consonant if the ratio of frequencies is super particular

$$(n + 1) : n \quad \forall n \in \mathbb{N}, n \geq 1 \tag{1.6}$$

like those found in (1.1), or one of their multiples. Given some empirical axioms, he tries to find the ratios usually attached to consonances in a deductive way. The problem is that some results are simpler than the basic axioms, like the consonance of the octave, see p. 29 of [Bai01].

Ptolemy (90–168) tried a less deductive approach and searched for a compromise between numbers and perception. He based his criteria of judgement on the following rule: reason (a mathematical foundation of the ratios, after the Pythagorean tradition) and empirical observation (in contrast to the Pythagoreans) must not conflict. In his *Harmonics*, he describes a classification of intervals that opened the set of consonances to multiples of the strict Pythagorean intervals.

$$\begin{aligned} \text{homophonic: } & 2^n : 1 \quad \forall n \geq 1 \\ \text{symphonic: } & \frac{3}{2} \cdot 2^n : 1 \quad \forall n \geq 0 \\ & \frac{4}{3} \cdot 2^n : 1 \quad \forall n \geq 0 \end{aligned} \tag{1.7}$$

Ptolemy has been criticised for mixing two incompatible approaches and his theory called syncretism. Nevertheless, it resolved the problem of the fourth added to an octave ($8:3$) that was missing in the Pythagorean system, while still being perceived as a consonance.

The Middle Ages

Musical techniques change. At the beginning of the 10th century, or perhaps even earlier, the ancient monodies were abandoned in favour of polyphonies, creating a demand for a new classification of consonances. Numbers lost the scientific and philosophical role they played during Antiquity. Their signification was reduced almost to symbols found in the Bible. Music theory progressively moved from the *quadrivium* with its cosmology and mathematics, towards the humanities of the *trivium*. Nevertheless, Pythagorean and Platonian thoughts were transmitted to the new Christian era with the help of Boethius (480–525) and his *De institutione musica*, whose main source was Nicomachus's lost *De Musica*.

While there was no conflict between Boethian theories and the early practice of *organum*, the consonance concept really evolved with the integration of new intervals. The thirds in particular, which were already considered to be *best* consonances in western England since the 13th century. Their first definition using super particular ratios goes back to the ancient Greeks such as Didymos, but Walter Odington, in his *Summa de Speculatione Musicae* dating from the 14th century, was the first one to use them as consonances in a polyphonic context.

$$\begin{aligned} \text{minor third: } & 6:5 \\ \text{major third: } & 5:4 \end{aligned} \tag{1.8}$$

The Renaissance

Gioseffo Zarlino integrated thirds and sixths into his theory by allowing further divisions of the monochord. The numbers 5 and 6 were added to the *senario*, the set of allowed numbers in ratios defining consonances as in (1.2), in order to accommodate for these new imperfect consonances [Zar58]. Note that number 6 is the smallest perfect number, i.e. a number equal to the sum of its prime factors.

$$1 \cdot 2 \cdot 3 = 1 + 2 + 3 \tag{1.9}$$

This leads to the definition of the major sixth with the help of a super particular ratio,

$$\text{major sixth: } 5:3 \tag{1.10}$$

but the minor sixth was still a problem, because the number 8 is necessary to define it. To circumvent this inconsistency, he defined it as a composition of a just fourth and a minor third.

$$\frac{8}{5} = \frac{4}{3} \cdot \frac{6}{5} \tag{1.11}$$

Integers bigger than 6 were taboo, especially the next prime number, the 7, that will bother theorists all the way up to Leonhard Euler. The number 8, necessary for the minor sixth is thus highly problematic.

Scientific Revolution in the 16th Century

The phenomenon of resonance had already been observed, but not formalised at that time. Rejecting the physical explanation of coincident beats supported by Isaac Beeckman, Johannes Kepler (1571–1630) produced the last model to explicitly rely on the *Harmony of spheres*. He was deeply influenced by Ptolemy's musical and astronomical theories, adopting his just scale and sharing his belief in cosmic (in his case divine) harmony. This can be considered as the culmination of his work, since it integrates his knowledge in astronomy, astrology and music, see Bruce Stephenson [Ste94] or Daniele Muzzulini [Muz06] for a more in-depth discussion.

Kepler's model tries to associate consonances with regular polygons [Kep19]. This purely geometrical approach uses the five Platonic solids to describe ratios found in the planetary orbits, and consonances. The circle is identified with a closed vibrating string divided equally by certain regular polygons. Such a model may be seen as an anticipation of the chromatic circle of the 20th-century musical set theory, see Sec. 2.1.1. Consonances consist only of ratios of parts and rests in polygons that are constructible geometrically: the diameter (seen as a 2-gon), the triangle, square, pentagon, hexagon, but not the heptagon. The number of faces is linked to the senario, which prohibits number 7. Three combinations form the ratios determining consonances and disso-

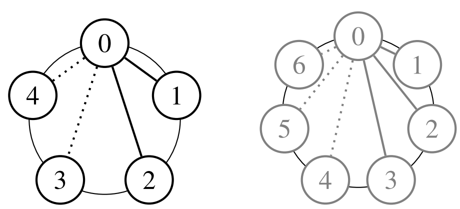

Figure 1.4: The constructible pentagon on the left serves for determining ratios for consonances. The first edges (clockwise) drawn with continuous lines determine the *consonant parts* ($p \in \{0, \dots, \lfloor \frac{n}{2} \rfloor\}$) the dotted lines are the *consonant rests* ($r = n - p$). The heptagon on the right is not constructible, its ratios determine the dissonances.

nances, see Fig. 1.4: part to total ($p : n$), part to rest ($p : r$), and rest to total ($r : n$). Using the regular pentagon we are able to construct most of the consonances:

$$\begin{array}{lll} p : r & 4 : 1 & \text{double octave} \\ & 3 : 2 & \text{fifth} \\ n : p & 5 : 1 & \text{major third augmented with a double octave} \\ & 5 : 2 & \text{major third augmented with a simple octave} \\ n : r & 5 : 4 & \text{major third} \\ & 5 : 3 & \text{major sixth} \end{array} \tag{1.12}$$

In order to generate a minor third ($6 : 5$), one then needs the hexagon. The octagon is necessary for the minor sixth ($8 : 5$). Inconsistencies in his system necessitate a trick to forbid the $7 : 1$ ratio introduced by the octagon in order to preserve the senario.

Another problem is that the 17-gon he categorised as a dissonant polygon will later be shown by Carl Friedrich Gauss to be constructible.

The Age of Enlightenment

One of the last attempts to explain consonances arithmetically is due to Leonhard Euler (1707–1783). The philosophical assumption that order lay at the foundation of taste and musical pleasure led him to develop a mathematical definition of an interval's sweetness, the *gradus suavitatis* Γ, see [Eul39] or Sec. B.3.1. of *ToM* [MGM02]. This function is expressed as

$$\Gamma(a) := 1 + \sum_{k=1}^{n} e_k \cdot (p_k - 1) \tag{1.13}$$

where the primes p_i and positive exponents e_i come from the decomposition of a into prime numbers: $a = \prod_{k=1}^{n} p_k^{e_k} \in \mathbb{N}$. The formula can be applied not only to intervals, but also to chords and their succession in time. Euler's system surpasses other classifications in versatility: it can be applied to entire pieces of music.

Intervals are no longer classified into discrete categories, but are ordered from the most consonant (the octave) to the most dissonant one (the major seventh) along a continuous scale. Since Euler was such a great physicist and contributed to the development of acoustics, he was also inspired by the coincidence of beats (see below), a physical justification for his model. Nevertheless, we will consider here his definition to be algebraic in nature.

Controversy on the Primary Role of Numbers

The philosophical debate on how the eternal and ideal world can participate within the temporal and material realm is as old as the controversy between Plato and his pupil Aristotle (384–322). The latter disagreed with the music of spheres in his *De Caelo*, 290b, where he speaks about silent, frictionless revolutions [Ari03]. Arguing that real things cannot be reduced to numbers, he criticises what he holds to be loose thinking and superstition in his *Metaphysics* [Ari07]. Most of the Greek treatises that survived follow the Pythagorean tradition, the only exception being the *Harmonics* of Aristoxenus [DR54], one of Aristotle's pupils. Aristoxenus emphasises the importance of physical facts and deductions based on experience. Auditory perception should always be trusted, to avoid errors originating from erroneous interpretations.

During the late Renaissance Vincenzo Galilei (1520–1591) studied antique theoretical treatises (Plutarch, Aristoxenus and Ptolemy) and already performed early acoustical experiments reproduced later by his son Galileo. He did not adhere to the faith of his former teacher Zarlino concerning mathematical foundations of *natural* rules in music theory. He expressed quite a modern opinion about consonances, refusing their limitation and maintaining that there is an infinity of them [Gal89].

The Age of Enlightenment saw Jean Le Rond d'Alembert (1717–1783) enthusiastically upholding the *Traité de l'harmonie* [Ram22] and restating Jean-Philippe Rameau's theory using a scientific formalism [d'A52]. But the second edition contains a foreword in which he clearly distances himself from Rameau:[8] d'Alembert chooses an experimental approach and opines that mathematical proofs cannot be established in music. Therefore deductions should prevail, as is the case in physics.

Finally, equal tuning, commonly used nowadays constitutes one of the main arguments against the pertinence of simple ratios. Most people seem to perfectly accustom themselves to consonances built on irrational proportions, which would have been inconceivable for a true Pythoagorean.[9]

1.4.2 Physical Models

It took a long time to understand the nature of sound as we do at the present time, meanwhile some observations and intuitions of the Greeks, who understood the analogy with sea waves, already agree with our contemporary scientific notions. Aristotle defined harmony as a technical theory of music and classified it as a physical science, and Ptolemy considered sound to be the state of after a percussion. This viewpoint was also shared by Vitruvius and Boethius, who quantifies sound by counting the number of impulses in the air in his *De institutione musica*. This tendency has been dominant during the whole development of the field. Despite the existence of a corpuscular interpretation, credited to Democritus (5th century BC) and Epicurus (4th century BC) and adopted by Pierre Gassendi (1592–1655) [Lin48], the debate never reached the importance it had in the field of optics or quantum mechanics.

The mathematical theory of sound propagation started with Isaac Newton (1642–1727) and his *Principia* [New87], and was continued during the next century by Euler, Joseph Louis Lagrange (1736–1813) and d'Alembert. At about the same time Joseph Sauveur (1653–1716) gave the contemporary appellation "acoustics" to the field. He conducted experiments on harmonics [Sau97], and imagined the first modern interpretation of beats, the two building blocks on which Hermann von Helmholtz based his future theory.

The insight physicists gained into the nature of sound as well as the growing importance of experimental methodology that developed between the 16th and 18th centuries also influenced music theory. Simple proportions were no more considered as the cause (the link between celestial, instrumental and human harmony providing a philosophical justification), but appeared rather as a consequence of a physical phenomenon: the coincidence of oscillations. Some Aristotelian thoughts having become popular again since, this trend also concords with the idea that music should imitate nature.

The Advent of Modern Phenomenology

One of the original contributions of Galileo Galilei (1564–1642) to science was a new combination of mathematics and experimentation:

[8] See also his letter, dating from 1762: *Lettre à Monsieur Rameau pour prouver que le corps sonore ne nous donne pas et ne peut pas nous donner par lui-même aucune idée des proportions (Letter to Mr Rameau to show that a resonating body does not and even cannot give us any idea of proportions).*

[9] Hippasus of Metapontum (5th century BC) was a Pythagorean philosopher who is sometimes credited with the discovery of the irrationality of $\sqrt{2}$, on sea, and to have died by drowning as a consequence, either committing suicide or killed by his shipmates. Note that a ratio of $\sqrt{2}$ corresponds to a tritone in equal tuning.

> Philosophy [i.e., physics] is written in this grand book—I mean the universe—which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometrical figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering around in a dark labyrinth.[10]

In some ways he follows the principles stated by Ptolemy. Both have studied music and astronomy, but Galileo didn't adhere to the music of spheres. Instead, he carried on the acoustical experiments already performed by his father, and shows a modern understanding of the acoustical phenomenon:

> Bellissima osservazione per poter distinguer ad una ad una le onde nate dal tremore del corpo che risuona, che son poi quelle che, diffuse per l'aria, vanno a far la titillazione su 'l timpano del nostro orecchio, la quale nell'anima ci doventa suono.[11]

His theory of intervals is based on a physical analogy between the vibrating string and the pendulum [Gal38].

Coincident Beats

Isaac Beeckman (1588–1637) influenced greatly his two pupils René Descartes (1596–1650) and Marin Mersenne (1588–1648). His theory of sound and coincident beats (*ictus* in Latin) gave a physical foundation to numerical arguments: simple ratios tend to increase the number of coincident beats, yiedling a measure of consonance [Bee34], see Fig. 1.5. Whether Beeckman and his followers pool the beats of both notes (as they should) or compute ratios while considering only the upper one (as they incorrectly do), is discussed more in depth in [Bai01], page 76 and following.

$m = 5$ ● · · ○ · · ○ · · ○ · · ○ · ·
$n = 3$ ● · · · · ○ · · · · ○ · · · ·

Figure 1.5: Isaac Beeckman's theory of coincident beats. Beats occur simultaneously only at the beginning of a cycle (●) and then alternate (○). The ratio of coincident beats is $(1+1)/(3+5) = 1/4$ while the computation for the upper note only yields $1/5$.

Beeckman's followers disagreed with him on the nature of sound. Beeckman conceives it to be corpuscular, whereas Descartes and Mersenne perceive it as an undulatory model.

Mersenne suggests two classifications in his *Traité de l'harmonie universelle*: the first one is based on the first harmonic, and orders intervals according to their sweetness to the ear, a concept which would be considerably developed by Rameau at a later date

[10] From *The Assayer* [Gal23]

[11] Waves are produced by the vibrations of a sonorous body, which spread through the air, bringing to the tympanum of the ear a stimulus which the mind interprets as sound.

[Mer27]. His second classification relies on the idea of simplicity of numerical ratios. On page 24 of the same book, talking about the unconscious arithmetic performed by the mind, he gives it a mechanical foundation. In his eyes, the ears count the beats of air.

> [...] On peut dire que l'ouïe n'est autre chose que le dénombrement des battements de l'air, soit que l'âme les compte sans que nous l'apercevions, ou qu'elle en sente le nombre qui la touche: car Platon croit qu'elle est un nombre harmonique.[12]

He constructs a ranking for dissonances, and is the first one to abandon the *senario*.

Resonance

Descartes anticipates Rameau's resonance theory in his early *Compendium Musicae* [Des50]. He uses experimental observations on resonance between the strings of a lute to explain the Greek ratios, and emphasises the importance of arithmetic ($a-b = b-c$) over geometric ($\frac{a}{b} = \frac{b}{c}$) ratios, thereby already invoking perceptual arguments:

> Ille proportio Arithmica esse debet, non Geometrica. Cujus ratio est, quia non tam multa in ea sunt advertenda, cum aequales sint ubique differentiae, ideoque non tantopere sensus fatigetur, ut omnia quae in ea sunt sitincte percipiat.[13]

Similar arguments are used for limiting the senario to the first six numbers:

> [...] Nec ulterius fit divisio, quia scilicet aurium imbecilitas sine laborer majores sonorum differentias non posset distinguere.[14]

Talking about resonance, we should mention again d'Alembert's scientific work on Rameau's treatise on harmony [d'A52].

Unsatisfactory Aspects of the Physical Approach

Interval classifications based on acoustical considerations do not give a clear distinction between consonances and dissonances. They are a continuous measure from one extreme to the other, wherein frontiers tend to vanish. Unable to build a theory free from contradiction, Mersenne and Descartes admitted that the distinction between consonance and dissonance is a fuzzy subject, as expressed in a letter from the latter to the former:

[12] One could say hearing is nothing but counting beats of the air, either that the soul counts them unconsciously, or that it feels the number that touches it: because Plato believes that it is a harmonic number.

[13] This ratio has to be arithmetical, not geometrical. The reason is that there are not so many things in the former to notice, since differences are always equal, so that the senses do not get tired of trying to perceive all the elements it contains.

[14] The divison [of the monochord] should not be carried on further, because, due to its weakness, the ear could not distinguish between more sound differences without effort.

> La douzième est plus simple que la quinte. Je dis plus simple, non pas plus agréable; car il faut remarquer que tout ce calcul sert seulement pour montrer quelles consonances sont les plus simples, ou si vous voulez, les plus douces et parfaites, mais non pas pour cela les plus agréables. Pour déterminer ce qui est plus agréable, il faut supposer la capacité de l'auditeur, laquelle change comme le goût, selon les personnes.[15]

1.4.3 Perceptual Models

Music may act on the soul through the senses. It had already been stated by Aristoxenus and was considered common sense since the Classical Age. A hundred years later, Helmholtz will lead the way to modern acoustics and research on auditory perception.

The 19th Century

Hermann von Helmholtz (1821–1894) was simultaneously an authority in physics, mathematics and medicine. He brought together the Fourier Decomposition of a complex sound into pure sine waves and the understanding of the operation of the inner ear's basilar membrane. These are essential building blocks for a theory of consonance based on interference between partials, see [Hel63]. Although imperfect, his classification confirms the habits and general consensus in music at this time, and like Euler's *gradus*, shows a continuum from consonances to dissonances, the latter being considered as bad consonances.

He considered the first nine harmonics of a violin sound $\{f_i|0 \leq i \leq 8\}$, and interference between partials being the least sweet for frequency difference $\Delta f_{ij} = f_i - f_j$ about 33 Hz. Note that the measure he computes depends on, and therefore takes into account the chosen timbre which is determined by the relative amplitudes of harmonics. Fig. 1.6 shows his continuous roughness function.

The 20th Century

Since Helmholtz's time, scientists have often endeavoured to gain insight into the working of the human ear, see for example *The science of musical sound* [Pie83] or p. 227–228 in *An Introduction to the Psychology of Hearing* [Moo08]. A refinement to Helmholtz's model has been done by Plomp and Levelt [PL65], who investigated the perception of pure tones. Other theories have been elaborated, relying on learning processes and pattern matching [Ter74], or on similarity patterns in neural discharge, see [Mey98] and [TCDB01].

Nevertheless, our understanding of how the brain handles auditory information remains very limited.

[15] The twelfth is simpler than the fifth. I say simpler, and not more pleasant; because one must note that this computation serves only to show which consonances are the simplest ones, I mean the sweetest and the most perfect ones, but not the most pleasant. In order to establish what is the most pleasant, one must assume the ability of the hearer, which, like taste, changes from one person to the other.

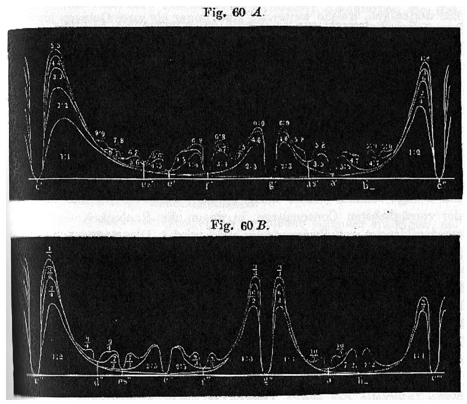

Figure 1.6: Helmholtz's continuous *Grad des Wohlklanges der Consonanzen* (consonance degree function), as figuring on page 303 in [Hel63]. Roughness caused by interfering harmonics of a violin versus (continuous) interval. The upper part (A) shows the intervals inside the first octave, the lower one (B) intervals inside the next one.

Open Questions in Perception Theory

It has been objected that Helmholtz-like theories took into account a change in timbre and overtones, while perception of consonances and dissonances does not vary with the instrument. Nor should overtone interference when separated sounds follow each other in time, like in melodies, or when both ears hear two different sounds, something that can easily be achieved with headphones.

Experiments also show that the perception of intervals varies with frequency and amplitude [SD48], and that the dissonant character of some intervals depends on non-linear effects producing rough combinations tones inside the ear.

In this field too, many hypotheses have been proposed, and nobody pretends to have a full understanding of what is happening in our ears and brain.

1.4.4 The Changing Status of the Fourth

The fourth constitutes the best and most famous illustration of the continuous evolution an interval could undergo during history. This was due to a concurrence by the thirds and the fifth, as well as to the development of new compositional techniques, especially counterpoint, see Sec. 1.5.

It will be shown in Sec. 2.1.3 that the fourth can be considered as a consonance for mathematical counterpoint. The fourth replaces the fifth, avoiding its undesirable proximity, and the generated set of rules of composition contain an interdiction against parallel movement of fourths.

Historically, the fourth began as a perfect consonance for the Greeks. It encompasses the tetrachord, which served as a basis of melody. Later, it was also favoured in the earliest forms of organum for parallel motion between two voices. Nevertheless, the combination of a fourth with an octave was considered to be dissonant by the

Pythagoreans, while the same combination with a fifth was still consonant.

A slow decadence begins in the 12th and 13th centuries with the advent of polyphonic music, and fifths gradually replaced the fourths. In the 15th century, the fourth is considered as a consonance only if it is located in the upper parts of vertical sonorities. At the end of the century, composers such as Dufay avoid it and Tinctoris eventually fixes its dissonant character in his treatise of 1473 [Tin73].

During the Renaissance, the fourth is thought of as being unstable (something between a third and a fifth), and consonant only if thought of as the inversion of a fifth.

At the beginning of the 17th century, due to the discrepancy between simplicity of ratio and musical usage, the fourth causes problems in both Mersenne's classifications, named Mersenne I respectively II in Tab. 1.2, and brings also some inconsistencies in Descartes's system. He complains about the fourth in [Des50]:

> Haec infelicissima est consonantiarum omnium, nec unquam in cantilenis adhibetur, nisi oer accidents & cum aliarum adiumento. Non quidem quod magis imperfecta sit, quam tertia minor aut secta; sed quia tam vicina est quintae, ut coram huius suavitate tota illius gratia evanescat.[16]

The rehabilitation of the fourth occurs in the 20th century. Chords built of perfect fourths are considered to be stable and are again extensively used, for example in Bartók's *Concerto for Orchestra*, Alexander Nikolayevich Scriabin's late works, or Arnold Schönberg's *Chamber Symphony* no 1, as well as in jazz and popular music—remember that the strings of a guitar are tuned a fourth apart.

Conclusion

All theories developed so far, whether on a mathematical, physical, or physiological basis, fail to describe consonances and dissonances by purely rational means. A conclusive and irrefutable explanation has never been established to explain their perceived *sweetness*, or a classification of the intervals.

The apprehension of consonances is a really complex phenomenon, involving physical, mathematical, medical, and psychological dimensions at least. Both *nature* and *culture* are concerned. To be unaware of only one of these aspects leads to a partial, unsatisfactory description.

> In psychoanalytic thinking, there is an effort to explore the passion in the mathematician's proof as well as an effort to use reason to understand the most primitive fantasy. The unconscious has its own highly structured language, which can be deciphered and analyzed. Logic has an affective side, an affect has a logic.[17]

The concept of dissonance has always been weaker than that of consonance. Dissonances are either defined as their complement, or as *bad* consonances. As Western

[16] This consonance is the most unfortunate of all. It is never used, except by accident or with the help of others. Not because the fourth is less perfect than the minor third or sixth, but it is so similar to the fifth that when compared, it completely loses its grace.

[17] Sherry Turkle, *Growing Up in the Age of Intelligent Machines: Reconstruction of the Psychological and Reconsiderations of the Human* in [Kur90], page 72.

music evolved, they disappeared progressively, swallowed up by an ever-growing set of admissible intervals that listeners had got used to. In their turn, consonances themselves were frowned upon within the framework of 20th century atonal music. One could note a parallel evolution of tuning from the mostly *monarchic* Pythagorean towards the totally *democratic* equal tuning used nowadays. An oversimplified but appealing image explains that the occurrence of this phenomenon is due to musicians progressively conquering higher harmonics.

As the knowledge about the physics of sound and the physiology of the human ear expanded—we may be about to build a theory integrating all aspects of the perception of intervals—this topic, once of primary interest for the ancient compositional techniques of the Renaissance's polyphonies, completely lost its relevance in the eyes of most of today's composers.

The generalisation of the consonance and dissonance concept that developed into the mathematical counterpoint model presented in Sec. 2.1.3 then takes another direction. Despite its Pythagorean flavour—it is essentially of algebraic essence, and it clearly separates intervals into two distinct classes—this model heavily relies on symmetry, a principle that plays a key role in the elaboration of laws in modern physics, e.g. particle physics. The originality of this theory is based upon its ability to open paths for shaping and composing new music. It provides a whole bunch of different choices for *consonances*, rather than giving an a posteriori description of an existing musical practice.

1.5 Treatises on Counterpoint

The question of consonances constitutes only a very small part of the rules governing counterpoint. The vast majority is concerned with the proper succession of intervals. Looking at the works treating this delicate question, one can see that they do not really present a theory, rather a description and codification of contemporary practice.

Most historical treatises were devoted to the polyphonic music of the Renaissance, but counterpoint is still taught at the conservatories—a student in composition needs to know the Classics. One of the treatises mostly used nowadays has been written during the first part of the 17th century: Fux's *Gradus ad Parnassum* was inspired by Palestrina's music, considered to be one of the purest examples of these compositional techniques. But the *Gradus* is not simply a manual of composition in Palestrina's style, nor does it describe his music exactly. Read Loris Azaroni's article for a comprehensive study of treatises and the pedagogical tradition of counterpoint [Aza06], or Tittel for a modern exposition of Fux's original work [Tit59]. The set of rules codified by Fux serves as a reference for mathematical counterpoint theory, see Sec. 31.4.1 in *ToM* [MGM02].

The Middle Ages

At the beginning, counterpoints were improvised. The name *discantus*, used to designate a specific kind of polyphonic music, appears to be related to the contrary motion of voices. But its meaning is very complex and changing. Centuries later, Johannes de

Table 1.2: Some of the historical consonance-dissonance dichotomies. Intervals are given in half-tones, above their frequency ratios, see [AM01] and [Bai01] for more details.

Period	Consonances - Dissonances													
Nicomachus (Ist century BC)														
	12	15	24	7	5				The rest is dissonant					
Pyth. Temp.	2 : 1	3 : 1	4 : 1	3 : 2	5 : 4									
Johannes de Garlandia (1241)														
	Perfect		Intermdeiate		Imperfect		Imperfect		Intermdeiate		Perfect			
	0	8	7	5	4	3	9	10	8	2	1	11		
Beeckman (1634)														
	12	7	5	4	3									
Descartes (1618)														
	12	7	4	5	9	3	8							
	2 : 1	3 : 2	5 : 4	4 : 3	5 : 3	6 : 5	8 : 5							
Mersenne I (1627)														
	12	7	4	3	9	8	5							
Just Temp.	2:1	3:2	5 : 4	6 : 5	5 : 3	8 : 5	4:3							
Mersenne II (1627)														
	12	7	5	9	4	3	8							
Beats/period	2	4	6	7	8	10	12							
Leibniz (1710)														
	12	9	8	7	5	4	3	2						
	2:1	5:3	8:5	3:2	4:3	5:4	6:5	9:8	10:9	16:15	25:24			
Euler (1739)														
	12	7	5	9	4	3	8	2	10+	10-	1	11		
	2:1	3:2	4:3	5:3	5:4	6:5	8:5	9:8	9:5	16:9	10:9	15:8		
Gradus suavitatis	II	IV	V	VII	VII	VIII	VIII	VIII	IX	IX	X	X		
Helmholtz (1863)														
	12	7	5	9	4	10	3	6	8	11	2	4+	1	
	2:1	3:2	4:3	5:3	5:4	7:4	6:5	7:5	8:5	9:5	8:7	9:7	9:8	
Partials friction	50	16.7	8.3	6.7	5.0	3.6	3.3	2.8	2.5	2.2	1.8	1.6	1.4	
Modern conception														
	Perfect			Imperfect			Mixte			Dissonances				
	12	7	9	8	4	3	5	11	10	6	2	1		
Equal Temp.	$2:1$	$2^{\frac{7}{12}}:1$	$2^{\frac{3}{4}}:1$	$2^{\frac{2}{3}}:1$	$2^{\frac{1}{3}}:1$	$2^{\frac{1}{4}}:1$	$2^{\frac{5}{12}}:1$	$2^{\frac{11}{12}}:1$	$2^{\frac{5}{6}}:1$	$2^{\frac{1}{2}}:1$	$2^{\frac{1}{6}}:1$	$2^{\frac{1}{12}}:1$		
Atonal conception, [Fal59]														
				Base							Prohibited			
	2	10	11	13	14	6	2	1	12	7	3	4	8	9

Garlandia opposed it to the *organum*, the name given to polyphony up to the 12th century, whose oldest treatises go back to the 9th century. See the *Musica enchiriadis et Scolica* written in the north of France around 860, Guido d'Arezzo's *Micrologus* from 1028, or the remaining texts from the School of Notre-Dame collected in the *Magnus liber organi de graduali et antiphonario pro servitio divino* at the beginning of the 13th century.

Philippe de Vitry may have been the author of the *Ars Nova*. The new compositional and notational techniques he introduced in 1322 allowed more rhythmical precision and complexity.

The Renaissance

Prosdocimo de Beldomandi mentions the interdiction of parallel octaves and fifths already in the *Tractatus de contrapuncto* from 1412. Johannes Tinctoris wrote the first corpus of stable rules in the *Liber de arte contrapuncti, Complexus effectuum musices* from 1477. The state of the art concerning methods and pedagogy used at the end of the 15th century can also be found in [Tin73], the first printed dictionary of music. Pietro Aaron's treatise dating from 1525 is one of the clearest before Zarlino [Aar25]. It contains a discussion about the emotional quality of minor and major chords. Vicente Lusitano describes improvised counterpoint and illustrates it with many musical examples [Lus53].

In the middle of the 16th century, Zarlino is the first to present a rationalisation of the consonances and the idea that the counterpoint's harmony emerges from unity in diversity [Zar58]. He was influenced by the music of Adrian Willaert and prior theoretic works such as those from Beldomandi, Tinctoris, and Nicola Vicentino [Vic55]. Two of his pupils also discuss counterpoint: Giovanni Maria Artusi [Art86] and Orazio Tigrini [Tig88].

Some theorists tend to organise their treatises with exercises of growing difficulty: Lodovico Zacconi [Zac92] who mentions already the five species in 1592, Girolamo Diruta in 1593 [Dir93, Dir25], followed by Adriano Banchieri [Ban10], Giovanni Maria Bonancini [Bon73], and Angelo Berardi [Ber81] during the 17th century. More and more complicated species will be accumulated, like in Berardi's work from 1687 [Ber87].

18th Century

Johann Joseph Fux, an Austrian composer, maestro di capella in Vienna and theorist wrote the *Gradus ad Parnassum* [Fux25], which has been used by Joseph Haydn, Wolfgang Amadeus Mozart, Ludwig van Beethoven and Franz Schubert. Fux popularised the classification of the counterpoint into five *species* of ascending difficulty. The methodic and rational organisation of the *Gradus* influenced all further manuals on the subject. Note that it was published three years after the *Traité de l'harmonie réduite à ses principes naturels* by Jean-Philippe Rameau [Ram22], a landmark in the evolution of harmony.

19th and 20th Centuries

Counterpoint lost its predominance with the rise of the tonal system. People started to think more and more *vertically*, but the old techniques were still taught in conservatories. Excepting some ambitious attempts to integrate and unify music theory, further treatises on counterpoint were mostly pedagogical refinements of the Fuxian work.

1.6 Some Applications

This dissertation proposes tools for computer-assisted composition in counterpoint worlds, not algorithms for composition. Its aim is to stimulate the curiosity of musicians to compose new music, unlike software which simulates an existing style. Nevertheless, since most of the research involving computers and counterpoint has been done in this domain, a short survey of the available literature seems necessary.

The corpus of compositional rules found in counterpoint is perhaps the most explicit in Western music. Its apparent ability to transfer seamlessly to software implementation constitutes an irresistible temptation for algorithmic composers and computer scientists.

A broad reference book such as *The Computer Music Tutorial* offers a concise introduction to this field [Roa96]. There are also more specialised books by Schottstaedt [Sch89], Cope [Cop91, Cop00], or even Kurzweil's introductory work focusing mainly on artificial intelligence [Kur90], which contains a chapter about music.

The oldest and most basic way to proceed is, paraphrasing Albert Einstein, to roll dices and keep only the lucky strikes, i.e. to let the computer filter out a randomly generated material, according to given constraints or optimisation criteria.

Lejaren Hiller's *Illiac Suite*, a string quartet randomly generated by a computer under contrapuntal constraints in 1959, is perhaps the most famous illustration of this idea, and constitutes the first attempt in history to make a computer composed music [HI59]. Farbood and Schoner also discuss stochastic methods [FS01], as well as Ames, a composer who integrates constraints and sorting methods [Ame83]. Another example using logic is given by Boenn et al. [BBDVF09], which apply answer set programming to automatic counterpoint composition.

Artifical intelligence appeared to be the right answer to the unsatisfactory results produced by oversimplified algorithms, see for example Widmer [Wid92] and Cope [Cop96]. Polito et al. used genetic algorithms for generating 16th-century counterpoints [PDBB97], as well as Acevedo for Palestrina-style counterpoint [Ace05], following the rules compiled by Jeppesen [Jep68]. Chemillier used finite state automata [Che90], and Adiloglu et al. neural networks [AA07]. Different approaches can be mixed as well, as do Goldman et al. [GGRL99] by mixing agents and neural networks. Another possibility is to use reinforcement learning [PA09].

These works are mentioned for their scientific or historic interest. However complex the algorithms involved, they do not seem to compensate for the lack of knowledge we have about human perception and music composition, and the music generated appears to be of poor interest. It is necessary to dig into the global structure, which is much harder to formalise. The counterpoint worlds of Chap. 3 could be seen as a proposition in that direction. However, as pointed out in Tittel's *Neuer Gradus*, the ultimate goal is not the rule, but the melody you write [Tit59].

Another application of computer science in counterpoint is computer-assisted composition and teaching. Newcomb wrote the *LASSO* software for teaching 16th-century counterpoint [New85], the *Counterpointer*[18] software treats species and free counterpoint, and Jones developed a *Counterpoint Assistant*[19] [Jon00].

1.7 Methodology

In Chap. 4, graph theory will be applied to represent counterpoints graphically and facilitate their manipulation. Its power of abstraction, versatility, huge corpus of algorithms and theoretical investigations, turns it into the tool of choice for carrying out this task.

We should first mention Leonhard Euler. He is considered as the father of graph theory for having formulated the Königsberg's bridges problem [Eul41]. His interest in music has already been discussed in Sec. 1.4, so he appears to be a pioneer in both fields this dissertation is mainly concerned with: mathematical music theory and graph theory.

Many books exist on this broad subject: West [Wes01] is a classic introduction, Harary [Har69] a well-established reference, and Diestel, available both in English or German, is also commendable [Die10]. Arrows being the essence of category theory, a more category theoretical flavour can be found in volume one of *Comprehensive Mathematics for Computer Scientists* [MMW04].

Some common graph algorithms are explained by Skienna [Ski98], who also gives a list of software packages and implementations.[20] Of particular interest is the discussion of data formats for graphs and the importance of incidence lists for improving efficiency on sparse graphs. Cormen's introduction is more detailed [CLRS09]. At the time of writing, a fourth volume of *The Art of Computer programming* is planned on this subject, but not available [Knu97]. It should become a valuable reference work once published. Goldman [GG08] discusses thoroughly implementation issues of abstract data types, including graphs. It also focuses on Java, the language in which our software is written.

There are many software packages implementing graphs. One of the most famous ones is perhaps *LEDA*[21] [MN99]. The *Boost*[22] C++ libraries also contain classes devoted to graph theory that may become part of the language's standard in the future. *JGraphT*[23] is an open-source Java project.

[18] Available at `http://www.ars-nova.com/`.

[19] Available at `http://arts.ucsc.edu/faculty/jones/`.

[20] Available from the Stony Brook Algorithm Repository at `http://www.cs.sunysb.edu/~algorith/files/graph-isomorphism.shtml`.

[21] A free version for Windows and Linux can be obtained from `http://www.algorithmic-solutions.com/leda/`, but the C++ source code must be asked for separately.

[22] Freely available at `http://www.boost.org/`.

[23] Available at `http://www.jgrapht.org/`.

Morphisms

On the general subject of graph homomorphisms, i.e. transformations of graphs preserving the arrow structure, a theoretical approach can be found in Hell and Nešetřil [HN04]. It shows that counting graph homomorphisms is an NP-complete problem, and that the complexity is P if the target graph has a loop or is bipartite. Dyer and Greenhill discuss this topic further [DG00], as well as Chen et al. [CTW08].

As is often the case, the available algorithms can be split into two categories: optimal algorithms which won't miss a solution but do not perform sufficiently fast, versus approximate heuristics which will perform faster but may miss the optimal solution. A third approach is to consider only graphs showing some particular properties, such as bounded orders, planarity, or trees, for which execution reduces to polynomial time.

The *isomorphic (sub-)graph problem* is a graph-matching problem[24] of great importance which found its first applications in mathematical chemistry (chemical structure) analysis). Its importance in such fields as artificial intelligence, especially computer vision, biological networks, semantic networks and data mining is continuously growing and led to the development of many new algorithms in the past decade. It encompasses classical graph theoretical problems like finding hamiltonian paths, shortest paths and cliques. Algorithms and theory have been reviewed by Bunke [Bun00], [BFG$^+$02], Raymond and Willett [RW02], [CFSV04], [JB08] or [LVG09].

The NP complexity has been discussed by Kobler et al. [KST93]. Backtracking procedures have been proposed by Schmidt et al., which performs in $O\left(n!n^3\right)$ [SD76], and by Berztiss who concentrated on directed graphs [Ber73]. Babai and Luks [BL83] developed an algorithm performing in $O\left(\sqrt{2^{n \log n}}\right)$. An approximate solution was given by Fischer and Matsliah who accepted some tolerance [FM06]. Kukluk et al. obtained an $O\left(n^2\right)$-time solution for planar graphs [KHC04]. Another strategy was found by Corneil and Gotlieb who proceed by reordering the vertices [CG70]. The most efficient package for automorphisms seems to be *Nauty*,[25] described by McKay in [McK81]. The *maximum common subgraph* isomorphism problem (MCS) is known to be NP-hard. Optimal solutions include several backtracking propositions. A reference algorithm performing in $O\left(n!n^3\right)$ has been invented by Ulmann [Ull76]. He was followed by McGregor [McG82], and Krissinel and Henrick [KH04]. The VF algorithm [CFCV04] has become a second, better performing reference, and is frequently used in benchmarks for testing newer algorithms. Many competitors have appeared in the last years. Often tailored for specific purposes, they can show better performance in special applications. As an alternative to search tree methods, let us also mention constraint programming [Rud00].

The search for the *strict* directed graphs morphisms explained in Sec. 4.1 is a subgraph isomorphism problem, known to be NP-complete. In Chap. 5, we detail a backtracking strategy exploiting a hierarchical structure to speed up the computation.

[24] It consists in identifying the vertices of two graphs in such a way that the connectivity structure is preserved.

[25] Available as a free download from `http://cs.anu.edu.au/~bdm/nauty/`.

Drawing

Another class of non-trivial problems found in graph theory is, beginning with the question of planarity, that of the graphical plane representation of an abstract graph. Di Battista et al. wrote a general book on the subject [DBETT98], followed by Kaufmann and Wagner [KW01] a few years later. Many resources can be found online, for example the web-site of the steering committee of the *Symposium on Graph Drawing*.[26] It contains a tutorial [CT94] and a link to the graph drawing e-print archive[27] among other things. Jünger and Mutzel surveyed software [JM04] and Di Battista the literature [DBETT94]. A summary of automatic graph drawing procedures can be found in Susan Sim's summary from which we took the automatic *spring* layout, which tries to minimise the energy of a physical system in which vertices are masses and edges springs [Sim96]. A powerful software package for graph representation and manipulation is called *Gephi*.[28]

1.8 Outline

The structure of the whole text is inspired by category theory: *Objects* are presented first, then come their morphisms, or transformations.

Chapter 2 summarises the mathematical procedure for simulating counterpoint: how to choose consonances (Sec. 2.1) and build the according system of compositional rules (Sec. 2.2). The original work can be found in Chap. 29 to 31 of *ToM* [MGM02], or Chap. 11 to 16 of [Maz07]. In addition, some parameters determining the shape of counterpoint worlds are discussed.

Structures relevant to systems of counterpoint rules, the counterpoint *worlds*, and requirements for their preservation by counterpoint world morphisms are formerly defined in Chap. 3. This will set a general framework for the next chapters.

A graph theoretical approach to the problem of transforming counterpoint worlds is presented in Chap. 4, where it will appear that all concepts previously defined find a natural translation into this abstract language. An algorithm for transforming counterpoint worlds is then described in detail in Chap. 5.

Chap. 6 introduces the functorial formalism underlying the software. Data exchange between different modules or plug-ins occurs via *forms* and *denotators*: a universal data formalism unique to Rubato and thoroughly described in *ToM* [MGM02] and [Mil06].

A user manual explaining how to operate the different *rubettes*, the software modules designed for counterpoint manipulation, is found in Chap. 7. The plug-ins implementing the material presented in Chap. 4 and 5 constitute a software suite for computer-assisted composition, and for playing with counterpoint rules.

Concluding remarks are presented in Chap. 8. Long tables and figures illustrating theoretical definitions for different octave divisions (macro-tonality) have been collected in the appendix.

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[26] Found at `http://graphdrawing.org/`.

[27] Available at `http://gdea.informatik.uni-koeln.de/`

[28] Its open-source code and Java binaries can be obtained at `http://gephi.org/`.

Chapter 2

Mathematical Counterpoint

Mazzola's counterpoint theory is an algebraical and geometrical procedure for generating a system of two-voiced note against note counterpoint rules. Given a set of consonant intervals on its input, it outputs the according table of their allowed and forbidden successions, i.e. it decides which are the legal steps that can be made in a counterpoint.

It was initially developed to simulate the rules of traditional Western counterpoint [Maz90]. The Fuxian rules are obtained with the usual consonances: thirds, fifth, sixths and octave. But many other choices fulfilling the algebraic requirements are possible. See Chap. 30 of *ToM* [MGM02].

This chapter follows Mazzola's original work without giving all the details and motivations behind the model. It only serves to introduce the algebraic constructions this dissertation relies on, and discuss some aspects relevant to the counterpoint worlds and graphs introduced in Chap. 3 resp. 4.

Sec. 2.1.1 introduces preliminary material such as the space in which notes and intervals live. How consonances can be chosen is explained in Sec. 2.1, and how the according tables ruling their succession can be generated in Sec. 2.2.

2.1 Choosing Dichotomies

Dichotomies are the segregation of the n intervals into two halves: $\frac{n}{2}$ consonances and $\frac{n}{2}$ dissonances, two sets that define the legal, respectively forbidden distances in semitones between cantus firmus and discantus.

In order to find the consonances admissible for the generation of counterpoint worlds, we will use a special subgroup of the affine group, symmetries that are also isometries on the torus of intervals. The contrapuntal symmetries in the next section will use the complete affine group.

2.1.1 The Circle Category

The model chosen provides a unified framework for the treatment of both notes and intervals, it is the *chromatic circle*, familiar to readers used to *musical* set theory. The same mathematical space serves for two distinct purposes. Such a distinction, between *intent* and *extent*[1], is expressed by the denotator formalism used in Sec. 6.4. The elements (pitch and interval classes) form a commutative ring. It is possible to act on them through affine transforms.

Chromatic Circle

The model assumes two conditions, almost natural in the Western musical tradition:

1. Octave equivalence. Two notes k_1 and k_2 belong to the same class if the ratio of their frequencies $\frac{\nu_2}{\nu_1}$ is an integer power of two.

$$\nu_2 \sim \nu_1 \Leftrightarrow \exists l \in \mathbb{Z} : \nu_2 = 2^l \cdot \nu_1 \tag{2.1}$$

2. Equal temperament and hence enharmony. The octave is divided into n equal intervals. The frequency $\nu_k, k \in \mathbb{Z}$ of the k-th note is defined relatively to the frequency ν_0 of a reference note 0. It is thus given by

$$\nu_k = 2^{\frac{k}{n}} \cdot \nu_0. \tag{2.2}$$

   The usual choice for a reference is $A3$, whose frequency ν_0 is equal to 440 Hz.

Conditions (2.1) and (2.2) allow us to reduce the (theoretically) infinite set of $k \in \mathbb{Z}$ notes to a finite set of n classes defined by the following equivalence relation:

$$k_1 \sim k_2 \Leftrightarrow k_1 - k_2 \in n\mathbb{Z} \tag{2.3}$$

Example 1. *The traditional denomination $A, B, C, \ldots, G$ expresses nothing else. It mentions only the pitch class, the information about the register gets lost and needs to be specified apart as in the common notation $A-2, \ldots, A2, A3, A4, \ldots, A7$.*

Such a class will be called a *pitch class*. And since intervals are measured as relative differences between two pitches, they obey the same algebra. This is justified by usage in classical counterpoint, where intervals are treated the same way whether they are simple or stacked on top of octaves. A notable exception is the unison and the octave itself.

Definition 1. *For an equal division of the octave into $n \in \mathbb{N}\backslash\{0\}$ notes or semitones, a* ***pitch*** *or* ***interval class*** *$[k]_n$ is a residual class modulo n.*

$$[k]_n := k + n \cdot \mathbb{Z} \quad k \in \mathbb{Z} \tag{2.4}$$

[1] See Sec. 6.2 in *ToM* [MGM02] or Sec. 3.1 in [Mil06]

These classes, the modular integers are supplied with addition and multiplication

$$\begin{aligned} [k]_n + [l]_n &= [k+l]_n \\ [k]_n \cdot [l]_n &= [k \cdot l]_n \end{aligned} \tag{2.5}$$

and form what is traditionally called the cyclic group $(\mathbb{Z}_n, +)$. It is abelian, and we will use it as a commutative ring throughout this dissertation.

Definition 2. *The* ***chromatic circle*** $\mathcal{C}_n$ *of* n *classes is the commutative ring*

$$\mathcal{C}_n := (\mathbb{Z}/n\mathbb{Z}, +, \cdot) \tag{2.6}$$

whose elements, the modular integers

$$[0]_n, \ldots, [n-1]_n \tag{2.7}$$

represent pitch or interval classes.

Example 2. *The usual division of the octave into* $n = 12$ *semitones gives the chromatic circle* $\mathcal{C}_{12}$ *shown in Fig. 2.1. It is usual to associate* $[0]$*, at noon, with the pitch class* C*. The next pitch classes follow clockwise in chromatic order:* $[1]$ *represents* $C\sharp$*,* $[2]$ *is* D*, and so on. The same holds for interval classes:* $[0]$ *corresponds to the unison (and also to the octave),* $[1]$ *to the minor second,* $[7]$ *to the fifth, etc. until the major seventh is reached at* $[11]$.

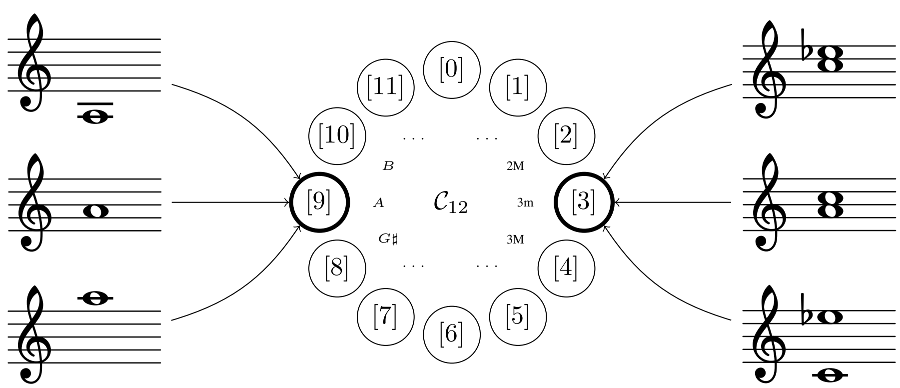


Figure 2.1: The chromatic circle in the 12 semitones context. It works exactly like a clock whose hours would be classes of pitches or intervals. Different representatives of the A class are shown on the left, and of the minor third class on the right.

Circle Symmetries

Equipped with addition and multiplication, the chromatic circle can be mapped into itself through affine transforms.

Definition 3. *A **circle symmetry** $\phi = e^b.u$ acting on a chromatic circle $\mathcal{C}_n$ is an affine automorphism of $\mathcal{C}_n$.*

$$\begin{aligned} \phi : \mathcal{C}_n &\longrightarrow \mathcal{C}_n \\ k &\longmapsto b + u \cdot k. \end{aligned} \tag{2.8}$$

The linear factor $[u]_n$ must belong to the units ${\mathcal{C}_n}^\times$ of $\mathcal{C}_n$, for ϕ to be invertible. The choice of the translation parameter $b \in \mathcal{C}_n$ is free.

Circle symmetries can be composed and their affine and bijective character is preserved.

Definition 4. *Together with the function composition $\circ$, the set of all circle symmetries of a chromatic circle $\mathcal{C}_n$ builds a group, the **affine group** $(\mathbb{A}_n, \circ)$:*

$$\mathbb{A}_n \cong \mathcal{C}_n \rtimes {\mathcal{C}_n}^\times \tag{2.9}$$

where $\rtimes$ stands for the semi-product. See theorem 29 on page 517 and Sec. C.3.2 in ToM *[MGM02].*

Example 3. *In the usual case, the units of $\mathcal{C}_{12}$ are ${\mathcal{C}_{12}}^\times = \{[1], [5], [7], [11]\}$. This yields a total amount of $\left|\mathbb{A}_{12}\right| = 4 \cdot 12 = 48$ circle symmetries. Their action on the chromatic circle is represented in Fig. 2.2 and 2.3.*

Circle Distance

The chromatic circle can further become a metric space once it is equipped with a distance measure. The simplest way to define one is to count the number of classes between two elements of $\mathcal{C}_n$, following the shortest arrow around the circle.

Definition 5. *The **circle distance** $\mathrm{d}_{\mathcal{C}_n}$ of a chromatic circle $\mathcal{C}_n$ is defined as the smallest amount of steps along $\mathcal{C}_n$ separating two classes.*

$$\begin{aligned} \mathrm{d}_{\mathcal{C}_n} : \mathcal{C}_n \times \mathcal{C}_n &\longrightarrow \mathbb{N} \\ (k, k') &\longmapsto \min_{i \in k, i' k'} |i - j| \end{aligned} \tag{2.10}$$

It can easily be verified that $\mathrm{d}_{\mathcal{C}_n}$ is indeed a distance, i.e. that it fulfils the three requirements of reflexivity, symmetry and transitivity. Values taken by $\mathrm{d}_{\mathcal{C}_n}$ lie between 0 and $\lfloor \frac{n}{2} \rfloor$.

Figures 2.2 and 2.3 show that shuffling classes by means of any symmetry whose linear factor is $u = [5]$ or $u = [7]$ does not preserve neighbourhood and distances. Thus, only the $2n$ members of the dihedral group

$$\mathbb{D}_n := \{e^b.u | u \in \{[-1]_n, [1]_n\}, b \in \mathcal{C}_n\} \tag{2.11}$$

a subgroup of $\mathbb{A}_n$ generated by transpositions and inversions, have the property to be an isometry on $\mathcal{C}_n$. The *interval torus* introduced in the next section will allow us to be less restrictive and extend the notion of isometry to the full $\mathbb{A}_{12}$.

The chromatic circles together with their affine automorphisms build the circle category $\mathfrak{Cir}$.

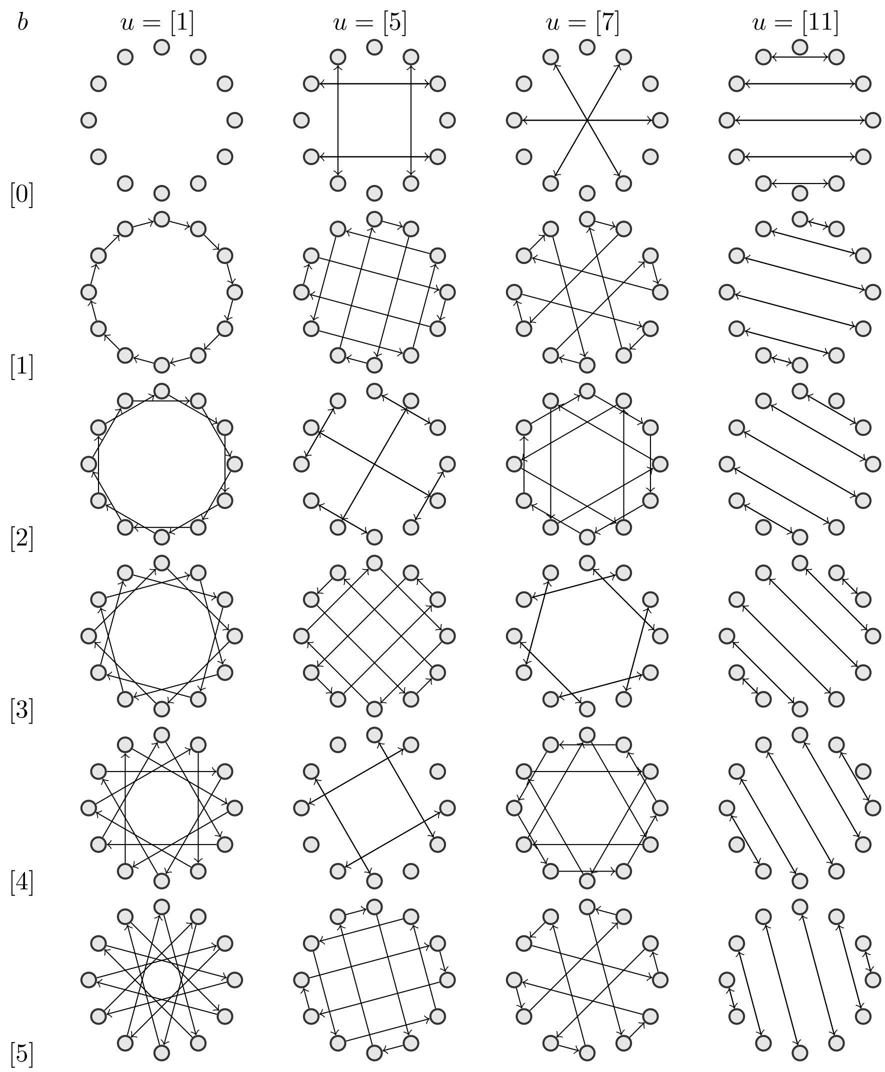


Figure 2.2: Action of the circle symmetries $e^b.u$ of $\mathcal{C}_{12}$. Transpositions b range from $[0]$ to $[5]$. Arrows indicate the trajectory along orbits. $[0]$ is located at twelve o'clock, classes follow clockwise.

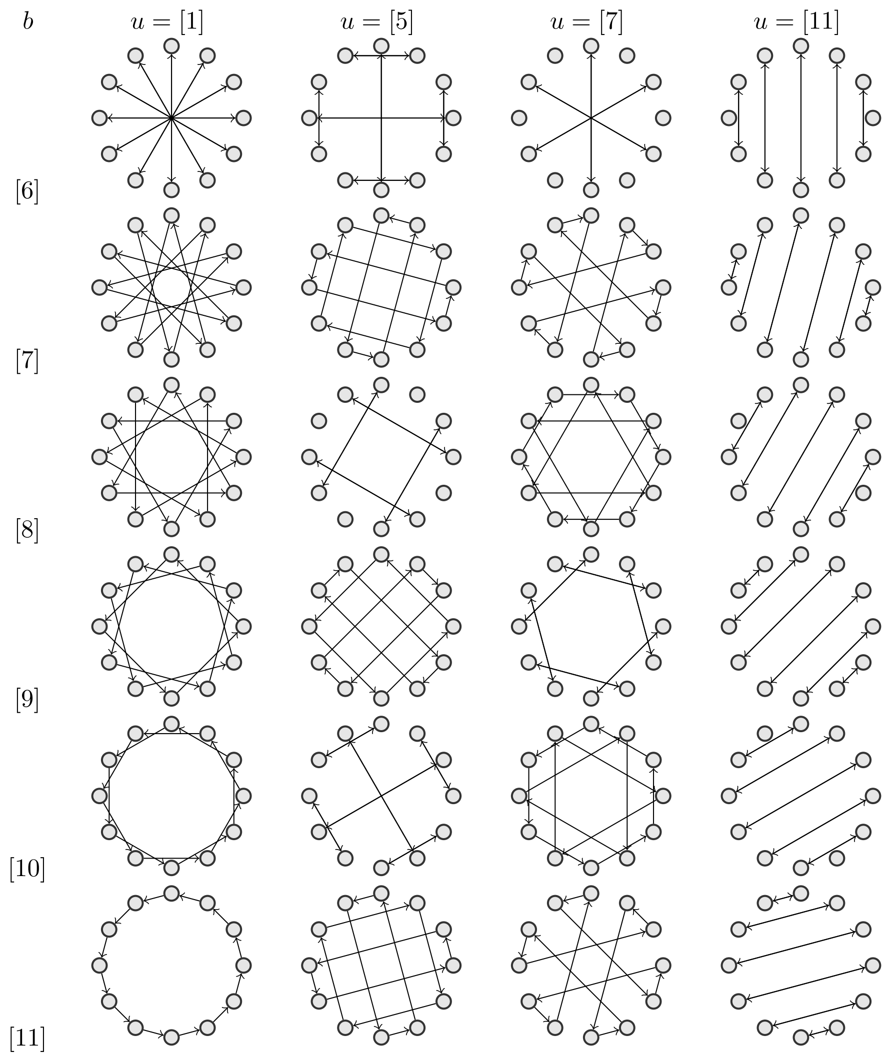


Figure 2.3: Action of the circle symmetries $e^b.u$ of $\mathcal{C}_{12}$. Transpositions b range from $[6]$ to $[11]$. Arrows indicate the trajectory along orbits. $[0]$ is located at twelve o'clock, classes follow clockwise.

2.1.2 The Torus Category

Because the integers form a principal ideal ring, they also constitue a *unique factorisation domain* (UFD). Modular integers inherit this property, which broadly means that a factorisation into irreducible—in this case also prime—elements is unique, see for example [Lan65].

$$\begin{aligned}
&\exists!\{p_1, \ldots, p_I\} \subseteq \mathbb{Z} \\
&\quad \wedge \exists!(r_i)_{i=1}^{I} \in {\mathbb{N}^*}^{I} \\
&\quad \text{such that } n = \prod_{i=1}^{I} p_i^{r_i}
\end{aligned} \tag{2.12}$$

where each p_i is prime. We will use this property to factor the chromatic circle and factorise it into an I-dimensional torus. The gain in dimension of the space of objects will leave more freedom for the morphisms for preserving distances.

Interval Torus

Pitch or interval classes get mapped from the discrete circle to the surface of a discrete torus.

Definition 6. *The* ***interval torus*** $\mathcal{T}_n$ *associated to a chromatic circle* $\mathcal{C}_n$ *is defined as its factorisation into* p*-Sylow subgroups* $\mathcal{C}_{p_i^{r_i}}$*, whose order are the greatest powers* r_i *of the prime numbers* p_i *that factor* n*.*

$$\mathcal{T}_n := \bigoplus_{i=1}^{I} \mathcal{C}_{p_i^{r_i}} \tag{2.13}$$

It is called *interval* torus since the factors $[p_i^{r_i}]_n$ all represent intervals building subcycles in the chromatic circle:

$$[1 \cdot p_i^{r_i}]_n, [2 \cdot p_i^{r_i}]_n, \ldots, [0]_n \tag{2.14}$$

Moving to the torus is straightforward

$$\begin{aligned}
t : \mathcal{C}_n &\longrightarrow \mathcal{T}_n \\
[k]_n &\longmapsto ([k]_{p_i^{r_i}})_{i=1}^{I}
\end{aligned} \tag{2.15}$$

but the way back is more tricky. A maximality condition

$$p_i^{r_i} | n \quad \text{and} \quad p_i^{r_i+1} \not| n \tag{2.16}$$

of the subgroups $\mathcal{C}_{p_i^{r_i}}$ is necessary to guarantee the bijectivity of the whole process

$$\mathcal{T}_n \cong \mathcal{C}_n \tag{2.17}$$

Under these conditions,

$$p_i^{r_i} + p_j^{r_j} \in {\mathcal{C}_n}^{\times} \quad \forall i \neq j \tag{2.18}$$

the Chinese remainder theorem applies, ensuring t is surjective. Its kernel is equal to

$$\mathrm{Ker}\,(t) = \cap_{i=1}^{I} (p_i^{r_i}) = 0. \tag{2.19}$$

It is therefore injective, hence bijective, so that the reverse operation t^{-1} is well defined, and can be computed by means of the extended Euclidian algorithm

$$\begin{aligned} t^{-1} : \mathcal{T}_n &\longrightarrow \mathcal{C}_n \\ \boldsymbol{k} = ([k_i]_{p_i^{r_i}})_{i=1}^{I} &\longmapsto [\sum_{i=1}^{I} s_i\,(\boldsymbol{k}) \cdot p_i^{r_i}]_n \end{aligned} \tag{2.20}$$

that delivers the set of factors $\{s_i\}_{i=1}^{I}$, see volume 2 of *The Art of Computer Programming* [Knu97]. The Chinese remainder theorem guarantees the uniqueness of the construction, since we factor n through different prime numbers p_i for which the greatest common divisor is $\langle p_1^{r_1}, \ldots, p_I^{r_I} \rangle = 1$.

Example 4. *In case of $n = 12$, the factorisation into primes gives $12 = 2^2 \cdot 3^1$. The torus becomes a two-dimensional* third *torus $\mathcal{T}_{12} = \mathcal{C}_4 \oplus \mathcal{C}_3$, see Fig. 2.4. The major third is represented by $[4]$ while $[3]$ is the minor third. The backward transformation is given by*

$$\begin{aligned} t^{-1} : \mathcal{T}_{12} &\longrightarrow \mathcal{C}_{12} \\ ([k_1]_4, [k_2]_3) &\longmapsto [-3 \cdot k_1 + 4 \cdot k_2] \end{aligned}$$

in which case the extended Euclidian algorithm delivers the factor set $s_1 = -3$ and $s_2 = 4$.

Torus distance

The torus can also be equipped with a distance that makes it a metric space. Such a measure is defined as the shortest path along the grid (horizontal and vertical steps in Fig. 2.4). Circle and torus being isomorphic by means of (2.15) and (2.20), this induces a new distance on the circle.

Definition 7. *The **torus distance** $\mathrm{d}_{\mathcal{T}_n}$ between two pitch class sets $[k]_n$ and $[k']_n$ on the chromatic circle $\mathcal{C}_n$ is defined as*

$$\begin{aligned} \mathrm{d}_{\mathcal{T}_n} : \mathcal{C}_n \times \mathcal{C}_n &\longrightarrow \mathbb{N} \\ ([k]_n, [k']_n) &\longmapsto \sum_{i=1}^{I} \mathrm{d}_{\mathcal{C}_{p_i^{r_i}}} \left([k]_{p_i^{r_i}}, [k']_{p_i^{r_i}} \right). \end{aligned} \tag{2.21}$$

$$\begin{tikzcd} \mathcal{C}_n \arrow[rr, tail, two heads, "t"] \arrow[dr, "\mathrm{d}_{\mathcal{T}_n}"'] & & \mathcal{T}_n \arrow[dl, "\times_i \mathrm{d}_{\mathcal{C}_{p_i^{r_i}}}"] \\ & \mathbb{N} & \end{tikzcd}$$

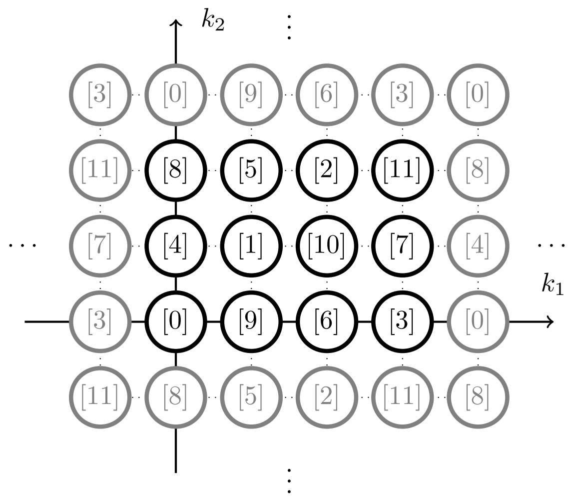


Figure 2.4: An elementary cell (in black) of the infinite, periodic grid representing the discrete torus $\mathcal{T}_{12}$. Periododic replications are shown in gray. A horizontal step to the right corresponds to a descent by a minor third, a vertical step up to an ascension by a major third.

It is the same principle as the *Manhattan* distance[2]. But because of the periodicity of the torus, one needs to check which direction is the shortest, like Columbus did: if one wants to travel to India, is it shorter by heading East or West?

Example 5. *On $\mathcal{T}_{12}$ the distance between two pitch classes denotes the minimal number of thirds (both minor and major) necessary to move from one to the other. For example, G lies two thirds apart from C: $\mathrm{d}_{\mathcal{T}_{12}}([0],[7]) = 2$, as shown in Fig. 2.5.*

Torus isometries

We will now search for all circle symmetries preserving distances on the torus.

Definition 8. *A **torus isometry** $\phi : \mathcal{C}_n \to \mathcal{C}_n$ is a circle symmetry that preserves distances on the interval torus $\mathcal{T}_n$.*

$$\mathrm{d}_{\mathcal{T}_n}(\phi(k), \phi(k')) = \mathrm{d}_{\mathcal{T}_n}(k, k') \quad \forall k, k' \in \mathcal{C}_n \tag{2.22}$$

The set of all torus isometries forms a subgroup of $\mathbb{A}_n$ and is written $\mathbb{I}_n$.

All circle symmetries of $\mathbb{A}_{12}$ are also torus isometries, see Fig. 2.6. This remarkable property is not true for all values of n. Fig. 29.1 on page 621 in *ToM* [MGM02] displays the torus as a donut in $\mathbb{R}^3$, in which each multiplication by a unit $u \in \mathbb{Z}_{12}{}^{\times}$ corresponds to a geometric transformation: mirrors and rotations around different planes and axes.

[2]This distance is measured in absolute distances along the coordinate axes, see [Wer07] for more details. The case $\mathbb{R}^2$ is is named after the map of Manhattan, where a walking distance from point $\boldsymbol{x}$ to $\boldsymbol{y}$ is measured in distances along horizontal streets and vertical avenues: $||\boldsymbol{x} - \boldsymbol{y}|| = \sum_{i=1}^{2} |x_i - y_i|$.

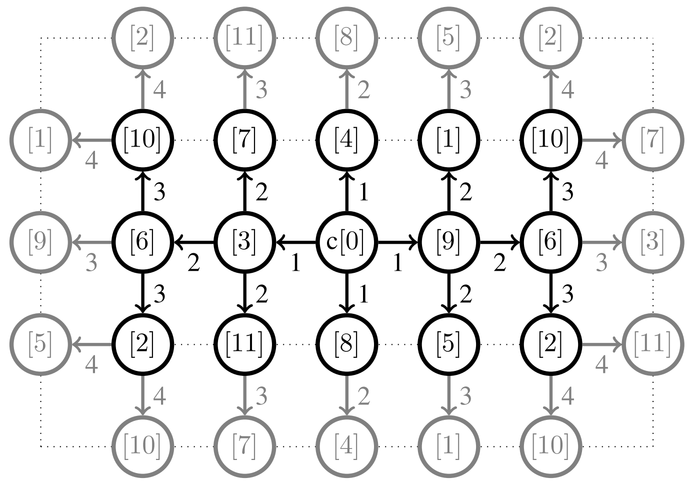


Figure 2.5: Distances on the third torus $\mathcal{T}_{12}$ from [0] are indicated next to the arrows. Shortest paths, represented by black lines, are not necessarily unique. The maximum distance is 3. "Distances" in gray show what happens if one does not take the shortest path.

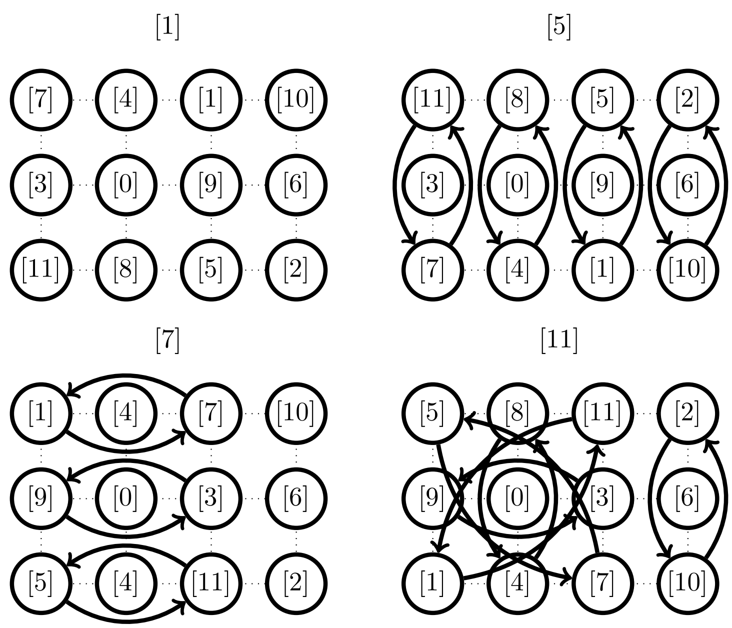


Figure 2.6: Multiplication by all four units $u \in \mathbb{Z}_{12}{}^{\times}$. Each one preserves the grid structure, thus is an isometry. Transpositions simply shift the whole grid and are naturally isometries

Examples and illustrations so far have concerned the usual case $n = 12$. We will now investigate the general case and show that 12 is indeed the maximum value n can take for $\mathbb{I}_n$ to equal $\mathbb{A}_n$. For $n > 12$ it will be smaller, possibly reducing to $\mathbb{D}_n$. This gives us less material for constructing dichotomies with symmetries in the next section. First, we establish an alternative description of a torus isometry.

Lemma 1. *A circle symmetry $\phi = e^b.u \in \mathbb{A}_n$ is a torus isometry iff its linear factor u is located one step away from the zero on each of the factor circles of the torus.*

$$\mathrm{d}_{\mathcal{C}_{p_i^{r_i}}}\left([0]_n, u\right) = 1 \quad \forall i : 1 \leq i \leq I \tag{2.23}$$

Proof. A translation e^b is always an isometry, and a composition of two isometries is still an isometry. Only the linear factor u needs to be investigated since it may shuffle elements of the torus.

- Condition (2.23) is sufficient. For any class $[k]_n \in \mathcal{C}_n$, it is guaranteed that ϕ belongs to the dihedral group of every factor circle $\mathcal{C}_{p_i^{r_i}}, 1 \leq i \leq I$, so the number of steps in any direction will not altered.

$$[u \cdot k]_{p_i^{r_i}} = [u]_{p_i^{r_i}} \cdot [k]_{p_i^{r_i}} = [1]_{p_i^{r_i}} \cdot [k]_{p_i^{r_i}} = [k]_{p_i^{r_i}} \tag{2.24}$$

  The circle symmetry ϕ is thus a torus isometry.

- Now we will verify that the condition (2.23) is necessary to make ϕ an isometry. The distance should be preserved for any pair of classes

$$\mathrm{d}_{\mathcal{T}_n}\left(\phi\left([k]_n\right), \phi\left([k']_n\right)\right) = \mathrm{d}_{\mathcal{T}_n}\left([k]_n, [k']_n\right) \quad \forall [k]_n, [k']_n \in \mathcal{C}_n \tag{2.25}$$

  This has to be true in particular for $z = [0]_n$ and $z' = [1]_n$. The right-hand side of (2.25) becomes:

$$\mathrm{d}_{\mathcal{T}_n}\left([0]_n, [1]_n\right) = \sum_{i=1}^{I} \mathrm{d}_{\mathcal{C}_{p_i^{r_i}}}\left([0]_{p_i^{r_i}}, [1]_{p_i^{r_i}}\right) = \sum_{i=1}^{I} 1 = I. \tag{2.26}$$

  The left-hand side of (2.25) corresponds to a multiplication by a unit u:

$$\begin{aligned}\mathrm{d}_{\mathcal{T}_n}\left(\phi\left([0]_n\right), \phi\left([1]_n\right)\right) &= \mathrm{d}_{\mathcal{T}_n}\left([0]_n, u\right) \\ &= \sum_{i=1}^{I} \mathrm{d}_{\mathcal{C}_{p_i^{r_i}}}\left([0]_{p_i^{r_i}}, [u]_{p_i^{r_i}}\right)\end{aligned} \tag{2.27}$$

  Since u is a unit, it is not a zero divisor, and $\langle u, n\rangle = 1$ holds. It is also coprime with every factor circle's dimension: $\langle u, p_i^{r_i}\rangle = 1$, so that the circle distance $\mathrm{d}_{\mathcal{C}_{p_i^{r_i}}}\left([0]_{p_i^{r_i}}, [u]_{p_i^{r_i}}\right)$ is strictly positive for every i. There are exactly I terms in the sum (2.27), so the only way not to exceed the sum I is for the distance in every dimension to equal one, which is exactly the requirement (2.23).

Condition (2.23) is hence equivalent to isometry. $\square$

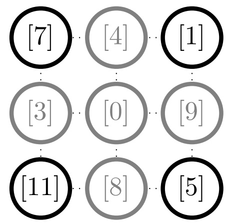


Figure 2.7: All four units of $\mathcal{C}_{12}$ sit around the origin on corners of a square, which is the $I = 2$-dimensional cube.

A geometric interpretation of condition (2.23) is that all units $u \in \mathcal{C}_n{}^{\times}$ have to lie at a distance I from the origin, which means they sit at the corners of an I-dimensional cube centred at the origin, see Fig. 2.7.

We are now able to enumerate all values of n for which all circle symmetries are isometries.

Corollary 1. *The only values of $n \in \mathbb{N}$ for which $\mathbb{I}_n = \mathbb{A}_n$ holds, are $2, 3, 4, 6$ and 12.*

Proof. The strategy of this proof is to count the units and check if there is enough place left for them on the corners of the cube previously mentioned. First, the number of units $\left|\mathcal{C}_n{}^{\times}\right|$ is given by Euler's totient function ϕ, that can be expressed in the following way:

$$\phi(n) = \prod_{i=1}^{I} (1 - p_i) \cdot p_i^{r_i - 1} \tag{2.28}$$

Then we must count the number of corners of the cube. Each dimension, i.e. each factor circle of the torus has room for two units: one projecting to $[u]_{p_i^{r_i}} = [-1]_{p_i^{r_i}}$ and the other one to $[u]_{p_i^{r_i}} = [+1]_{p_i^{r_i}}$, except if -1 and $+1$ collapse into the same class, which only happens in $\mathcal{C}_{2^1}$. Thus, the case $2|n$ but not $4|n$ needs to be handled separately, since there will be one dimension of the torus in which only one unit can take place, not two. Condition (2.23) can be fulfilled only if the $\phi(n/2)$ units can take place on 2^I corners. Both of the previous counts are the product of I factors, so the following inequality must hold for every factor:

$$(1 - p_i) \cdot p_i^{r_i - 1} \leq 2 \quad \forall i \in \{1, \ldots, I\}. \tag{2.29}$$

Thus, the only possible values for p and r are:

p	r	$(1-p)\cdot p^{r-1}$	
2	1	1	
	2	2	
	3	4	>2
	...		
3	1	2	
	2	6	>2
	...		
5	1	4	>2
...			

Possible values of n are the products consisting of all possible combinations of factors that satisfy (2.29):

$$n \in \{2^1, 3^1, 2^2, 2^1 \cdot 3^1, 2^2 \cdot 3^1\}. \tag{2.30}$$

□

As can be seen from corollary 1, the number of units $\phi(n)$ grows faster with n than the number of dimensions I added to the torus. The greatest possible value of n is 12, which could be considered as an optimal case: great enough to allow a two-dimensional torus (not a 1-dimensional circle as would be the case for any prime n) and small enough to keep all its units grouped around the origin. Tab. 2.1 shows what happens in other cases

Table 2.1: Cardinalities of the torus isometry groups $\mathbb{I}_n$. Number of semitones n, only even numbers are shown in order to allow a dichotomy. Dimension I of the interval torus $\mathcal{T}_n$, cardinalities of the isometry group $\mathbb{I}_n$ and the affine group $\mathbb{A}_n$. Only values of n up to 60 for which $|\mathbb{D}_n| < |\mathbb{I}_n|$ are shown.

n	I	$\lvert\mathbb{D}_n\rvert$	$\lvert\mathbb{I}_n\rvert$	$\lvert\mathbb{A}_n\rvert$
12	2	24	48	48
20	2	40	80	160
24	2	48	96	192
28	2	56	112	336
30	3	60	120	240
36	2	72	144	432
40	2	80	160	640
42	3	84	168	504
44	2	88	176	880
48	2	96	192	768
52	2	104	208	1248
56	2	112	224	1344
60	3	120	480	960

The chromatic circles together with the torus isometries build the torus category $\mathfrak{Tor}$.

2.1.3 Strong Dichotomies

The consonant or disonant character of an interval will not be chosen according to psychological or acoustic criteria, but rather through analysis of combinatorial and compositional considerations. In order to avoid choosing individual intervals, the set of consonances is considered and defined as a whole. There is no measure or ranking among consonances, but a pairing of each consonance with a dissonance through an *autocomplementary function* p_Δ. This gives a strong foundation to dissonances as a mirror, or dual world, to consonances.

Therefore, the set of available intervals $\mathcal{C}_n$ is divided into two equal parts, one consonant and the other dissonant. This section describes the way such a division can be done: The symmetries (torus isometries) of each possible set of consonances is investigated for the existence and uniqueness of such a mirror function. The fulfilment of these requirements makes it a candidate for a consonance/dissonance dichotomy. A further requirement, not always fulfillable, is that the consonances form a multiplicative monoid after multiplication with an appropriate unit.

Definition 9. *A **strong dichotomy** Δ of $\mathcal{C}_n$ is a partition of $\mathcal{C}_n$ into two subsets, the **consonances** K and the **dissonances** D of equal cardinality, for which there exists a unique **autocomplementary function** or **polarity function** $p_\Delta = e^r.w \in \mathbb{I}_n$, that maps consonances into dissonances and vice-versa:*

$$\begin{aligned} p_\Delta(K) &= D \\ p_\Delta(D) &= K. \end{aligned} \tag{2.31}$$

Definition 10. *A **multiplicative strong dichotomy** Δ of $\mathcal{C}_n$ is a strong dichotomy whose consonances K form a multiplicative monoid under multiplication by a suitable unit, i.e. $\exists u \in {\mathcal{C}_n}^\times$ such that:*

1. *It contains the neutral element:*

$$[1]_n \in u \cdot K \tag{2.32}$$

2. *It is closed under multiplication:*

$$k \cdot k' \in u \cdot K \quad \forall k, k' \in u \cdot K. \tag{2.33}$$

A strong dichotomy Δ designates a single pitch class set. A whole equivalence class can be defined by the action of the group of torus isometries $\mathbb{I}_n$ on the pitch class sets of cardinality $\frac{n}{2}$, that will be written $[\Delta]_{\mathbb{I}_n}$:

$$\Delta \sim_{\mathbb{I}_n} \Delta' :\Leftrightarrow \exists \phi \in \mathbb{I}_n : K' = \phi(K). \tag{2.34}$$

For $n = 12$, there are six strong dichotomy classes, each containing 48 dichotomies, for a total of 288 sets having a unique polarity function, see Fig. 2.9.

While a dichotomy Δ uniquely defines its polarity function p_Δ, the reverse is not true. Tab. 2.2 shows how different dichotomies may share the same polarity function, as is the case for $e^{[11]}.[11]$ which belongs to dichotomies of both classes 64 and 77.

Table 2.2: The 34 possible dichotomy classes of $\mathcal{C}_{12}$, in alphabetical order. The index of each dichotomy class $[\Delta]_{\mathbb{I}_{12}}$, as used in the *BollyWorld rubette* presented in Sec. 7.2.4. A star indicates a strong dichotomy. For comparison, the indices of chord classes given in appendix L of *ToM* [MGM02] are shown under *ToM*. A hat indicates the complement set. The consonance set K of the first representative Δ where consonances appear in black, dissonances in white, the autocomplementary function p_Δ and the internal symmetries $\phi : \phi(K) = K$ (for readability, the identity function $e^0.1$ is omitted). Only six of them are *strong*, i.e. eligible for building counterpoints. They show a polarity function (non-empty p_Δ column) and no internal symmetries (empty $\{\phi\}$ column). Numbers in the two last columns are residual classes modulo 12.

$[\Delta]_{\mathbb{I}_{12}}$	*ToM*	$\mathbb{1}_K$	p_Δ	$\{\phi\}$
63	63	●●●●●●○○○○○○	$e^6.1, e^{11}.11$	$e^5.11$
⋆64	64	●●●●●○●○○○○○	$e^{11}.11$	
65	65	●●●●●○○●○○○○		
66	66	●●●●●○○○●○○○		$e^4.11$
67	$\widehat{65}$	●●●●○●●○○○○○		
68	67	●●●●○●○●○○○○	$e^9.7, e^{11}.11$	$e^2.5$
⋆69	68	●●●●○●○○●○○○	$e^6.5$	
70	69	●●●●○●○○○●○○		
71	70	●●●●○●○○○○●○	$e^6.1, e^6.5, e^9.7, e^9.11$	$e^0.5, e^3.7, e^3.11$
⋆72	71	●●●●○○●●○○○○	$e^{11}.11$	
73	72	●●●●○○●○●○○○		$e^6.7$
74	73	●●●●○○●○○●○○		$e^3.11$
75	74	●●●●○○○●●○○○	$e^6.5, e^9.7$	$e^3.11$
76	$\widehat{66}$	●●●○●●●○○○○○		$e^6.11$
⋆77	75	●●●○●●○○●○○○	$e^{11}.11$	
78	76	●●●○●●○○○●○○	$e^6.1, e^{10}.5$	$e^4.5$
79	$\widehat{69}$	●●●○●●○○○○●○		
80	$\widehat{72}$	●●●○●○●●○○○○		$e^0.7$
81	77	●●●○●○●○●○○○	$e^3.7, e^{11}.11$	$e^8.5$
⋆82	78	●●●○●○●○○○●○	$e^9.11$	
83	79	●●●○●○○●●○○○		$e^0.7$
84	80	●●●○●○○●○○●○		$e^2.5, e^0.7, e^2.11$
85	81	●●●○●○○○●●○○	$e^3.7, e^7.11$	$e^4.5$
86	$\widehat{79}$	●●●○○●●○●○○○		$e^6.7$
⋆87	82	●●●○○●●○○●○○	$e^{10}.5$	
88	83	●●●○○○●●●○○○	$e^{3+6\mathbb{Z}}.\{1,7\}, e^{5+6\mathbb{Z}}.\{5,11\}$	$e^{6\mathbb{Z}}.\{1,7\}, e^{2+6\mathbb{Z}}.\{5,11\}$
89	$\widehat{73}$	●●○●●○●●○○○○		$e^7.11$
90	84	●●○●●○●○○●○○	$e^2.5, e^{11}.11$	$e^9.7$
91	$\widehat{80}$	●●○●●○●○○○●○		$e^{10}.5, e^6.7, e^4.11$
92	85	●●○●●○○●○●○○		$e^4.5, e^0.7, e^4.11$
93	$\widehat{85}$	●●○●○●●○○●○○		$e^0.5, e^6.7, e^6.11$
94	86	●●○●○○●●○●○○	$e^55, e^{11}5, e^511, e^{11}11$	$e^6.1, e^0.7, e^6.7$
95	87	●●○○●●○○●●○○	$e^{2+4\mathbb{Z}}.\{1,5\}, e^{3+4\mathbb{Z}}.\{7,11\}$	$e^{4\mathbb{Z}}.\{1,5\}, e^{1+4\mathbb{Z}}.\{7,11\}$
96	88	●○●○●○●○●○●○	$e^{1+2\mathbb{Z}}.\{{\mathcal{C}_{12}}^\times$	$e^{2\mathbb{Z}}.{\mathcal{C}_{12}}^\times$

Consonances and dissonances are perfectly symmetric: they are swapped by the polarity function. Choosing which ones should be considered as consonances, i.e. allowed intervals, is perfectly arbitrary as far as only definition 9 is involved. It is merely a matter of choosing different members inside a same class. The additional requirements contained in definition 10 breaks this symmetry by choosing the only one set having this multiplicative property.

Example 6. *The traditional Fuxian consonances*

$$K = \{[0], [3], [4], [7], [8], [9]\} \tag{2.35}$$

build a strong dichotomy. Its polarity function is $p_\Delta = e^{[2]}.[5]$*, as shown in Fig. 2.8.*

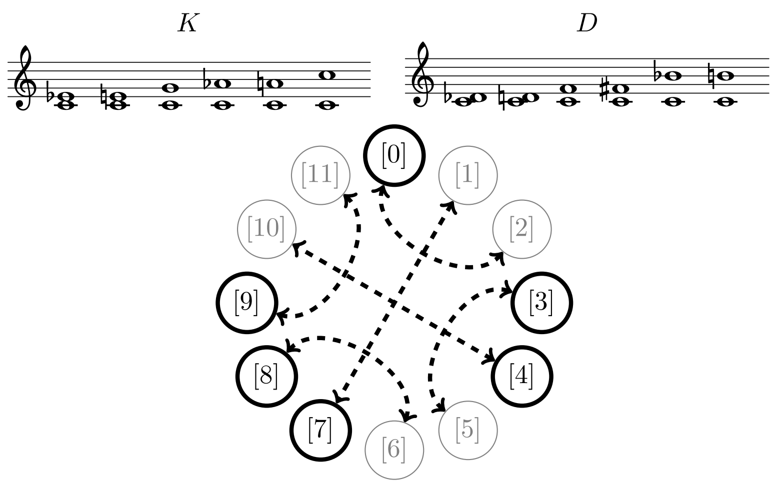


Figure 2.8: The Fuxian strong dichotomy. Consonances K are displayed in black, dissonances D in gray. Arrows show how the polarity function $ACFunction\Delta$ swaps consonance-dissonance pairs.

Autocomplementary Functions

Finally, we should highlight some constraints that affect the values that parameters of the autocomplementary function can take and which exceed the simple condition of w being a unit. The autocomplementary function must be unique, so it has to be its own inverse:

$$p_\Delta \circ p_\Delta = Id \tag{2.36}$$

Restrictions on the translation parameter r and the linear factor w follow:

$$\begin{aligned} r \cdot ([1]_n + w) &= [0]_n \\ w \cdot w &= [1]_n \end{aligned} \tag{2.37}$$

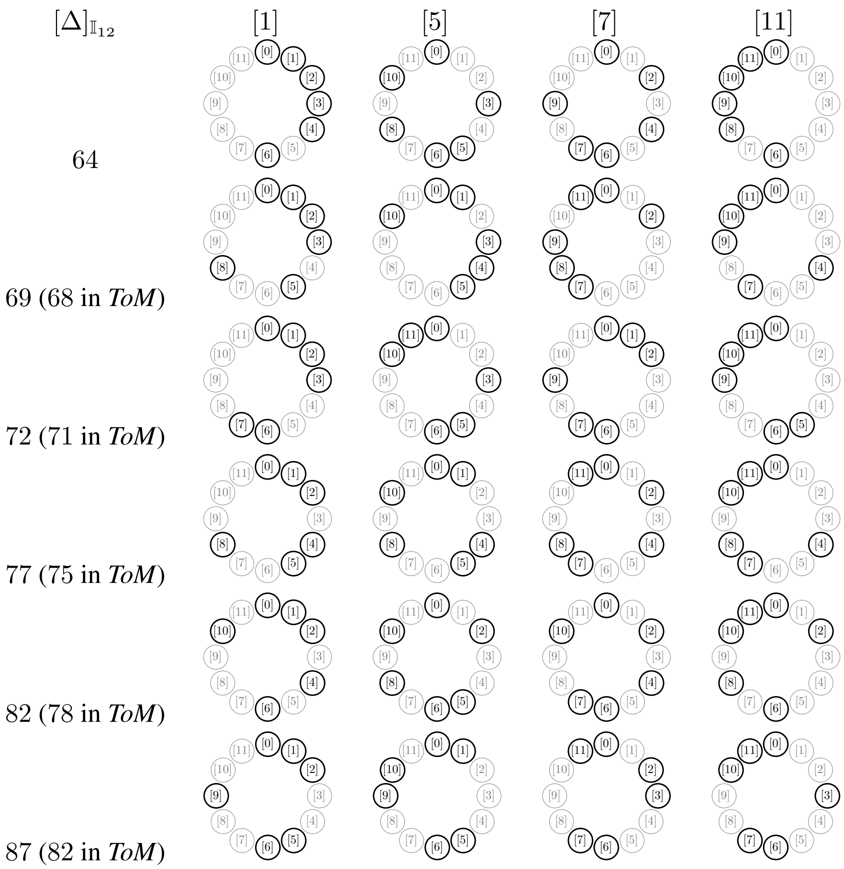


Figure 2.9: The six classes of strong dichotomies in $\mathcal{C}_{12}$. Only the multiplication of the first representative of Tab. 2.2 with each unit u of $\mathcal{C}_{12}$ is shown here. Each one should be transposed 12 times to obtain the full set of 48 dichotomies of a class.

This condition is not equivalent to the requirement of being an isometry of the interval torus. While

$$\mathrm{d}_{\mathcal{T}_n}\left([0]_n, w\right) = \dim \mathcal{T}_n \Longrightarrow \mathrm{d}_{\mathcal{C}_n}\left([0]_n, w\right) = 1, \tag{2.38}$$

the reciprocal is not true. Take for example $n = 5$ and $u = [2]_5 \in {\mathcal{C}_5}^{\times}$. $\mathrm{d}_{\mathcal{C}_5}\left([2^2]_5\right) = 1$ but $\mathrm{d}_{\mathcal{C}_5}\left([2]_5\right) = 2$. The same happens for $n = 24$ and $u = [5]_{24}$. This means that not all units yielding an isomorphism can serve for an autocomplementary function. Once again, $\mathcal{C}_{12}$ is an exception to this rule, since $[5^2] = [1]$ and $[7^2] = [1]$.

Multiplicative monoids

Example 7. *Multiplied by* $[7]$*, the classical fuxian consonances* K *yield a monoid.*

$$[7] \cdot K = \{[0], [1], [3], [4], [8], [9]\} \tag{2.39}$$

Only the the second half of the six classes $[\Delta]_{\mathbb{I}_{12}}$ contain dichotomies showing this property, see Tab. 2.3. Class 82 even shows two consonance sets K per factor u. Class 87 contains the Fuxian consonances. Tab. 2.3 shows the equivalent behaviour of minor seconds $[1]$, fourths $[5]$, fifths $[7]$ and major sevenths $[11]$ concerning monoids. Notice a special property shared by all units of $\mathcal{C}_{12}$: raised to its square, it gives $[1]$. Since the $[1]$ is mandatory to form a monoid, it appears systematically on each line of Tab. 2.3.

Table 2.3: Multiplicative monoids in $\mathcal{C}_{12}$. The dichotomy class index $[\Delta]_{\mathbb{I}_{12}}$ (numbers correspond to 75, 78 respectively 82 in *ToM*), and the unit u, necessary for transforming the consonance set K into a multiplicative monoid.

$[\Delta]_{\mathbb{I}_{12}}$	$[u]$	K
77	$[\mathbf{1}]$	$\{[0], [\mathbf{1}], [4], [6], [8], [9]\}$
	$[\mathbf{5}]$	$\{[0], [4], [\mathbf{5}], [6], [8], [9]\}$
	$[\mathbf{7}]$	$\{[0], [3], [4], [6], [\mathbf{7}], [8]\}$
	$[\mathbf{11}]$	$\{[0], [3], [4], [6], [8], [\mathbf{11}]\}$
82	$[\mathbf{1}]$	$\{[0], [\mathbf{1}], [4], [6], [8], [10]\}$
	$[\mathbf{5}]$	$\{[0], [2], [4], [\mathbf{5}], [6], [8]\}$
	$[\mathbf{7}]$	$\{[0], [4], [6], [\mathbf{7}], [8], [10]\}$
	$[\mathbf{11}]$	$\{[0], [2], [4], [6], [8], [\mathbf{11}]\}$
	$[\mathbf{1}]$	$\{[0], [\mathbf{1}], [2], [4], [8], [10]\}$
	$[\mathbf{5}]$	$\{[0], [2], [4], [\mathbf{5}], [8], [10]\}$
	$[\mathbf{7}]$	$\{[0], [2], [4], [\mathbf{7}], [8], [10]\}$
	$[\mathbf{11}]$	$\{[0], [2], [4], [8], [10], [\mathbf{11}]\}$
87	$[\mathbf{1}]$	$\{[0], [\mathbf{1}], [3], [4], [8], [9]\}$
	$[\mathbf{5}]$	$\{[0], [3], [4], [\mathbf{5}], [8], [9]\}$
	$[\mathbf{7}]$	$\{[0], [3], [4], [\mathbf{7}], [8], [9]\}$
	$[\mathbf{11}]$	$\{[0], [3], [4], [8], [9], [\mathbf{11}]\}$

Classification

All strong dichotomies and hence all counterpoint systems can be classified systematically, by sorting the pitch class sets linearly. The follwoing triple completely identifies a strong dichotomy:

1. The chromatic gamut size $n \in 2\mathbb{N}$ of the enclosing chromatic circle.

2. The index of the affine class, in alphabetical order of the indicator vectors shown in the third column of Tab. 2.2.

3. The index of the member of the affine class in alphabetical order, in alphabetical order of their respective indicator vectors.

Example 8. *The six strong dichotomies mentioned in* ToM *[MGM02] are completely identified by the triples* $(12, 64, 40)$*,* $(12, 69, 0)$*,* $(12, 72, 0)$*,* $(12, 77, 0)$*,*$(12, 82, 0)$ *and* $(12, 87, 17)$ *if we follow the numbering of Tab. 2.2 for dichotomy classes. Beware of the indices: they differ from the class numbers used in* ToM *[MGM02].*

Since they will be used extensively throughout this dissertation, we will simplify the notation when referring to the six examples in $\mathcal{C}_{12}$, and write Δ_{64}, Δ_{69}, Δ_{72}, Δ_{77}, Δ_{82} and Δ_{87}. Without any further specification, we will omit the chromatic gamut size when it is clear from context, and implicitly take as representative of a class its first member in alphabetical order (which is not the case for the Fuxian Δ_{64} and Dur Δ_{87} worlds, that have been chosen for their significance in musical tradition).

2.2 Constructing Interdiction Tables

We now turn to the heart of Mazzola's mathematical model: given a set of allowed intervals K, defined as the consonances of a strong dichotomy Δ, the algorithm has to decide which steps between pairs of successive consonances are allowed. A thorough presentation of the model with all its motivations and implications can be found in part VII of *ToM* [MGM02] or part III of [Maz07]. Only the algebraic aspects relevant to the shape of counterpoint worlds are discussed here.

2.2.1 Contrapuntal Intervals

The chromatic circle $\mathcal{C}_n$ serves for modelling single pitches or intervals, but it is not sufficient to describe two-voiced counterpoints. We need to distinguish between the two voices. Cantus firmus and discantus have a different function and origin: the discantus is added—usually afterwards—to the cantus firmus. This sequential enrichment can be thought of as a variation of the latter, even as an infinitesimal variation ε. Since the two voices may cross, we have to keep track of this asymmetry and cannot simply use the *set-theoretic* description of an interval, namely an unordered set containing the pitch classes belonging to each voice. A pair of pitch classes does not work either, since counterpoint rules focus on intervals and not individual pitches of the voices.

Dual Space

The model relies on the dual numbers of algebraic geometry, see for example [Har77]. Algebraically speaking, they are a quotient space of the ring of polynomials. On the geometrical side, they represent a tangent space (infinitesimal variation). For our purpose, we will consider them as a pair of numbers, one describing the anchor note, or cantus firmus, the second the interval from cantus firmus to discantus. See Sec. 29.2 in *ToM* [MGM02] for more details.

Definition 11. *The space of **dual numbers** $\mathcal{C}_n[\varepsilon]$, also called **dual space**, is defined as the ring of polynomials $\mathbb{Z}_n[\varepsilon]$ with coefficients in $\mathbb{Z}_n$, factored with the ideal generated by the polynomial of second order ε^2:*

$$\mathcal{C}_n[\varepsilon] := \mathbb{Z}_n[\varepsilon]/(\varepsilon^2). \tag{2.40}$$

A dual number is written $x + \varepsilon.y, x, y \in \mathbb{Z}_n$. Addition and multiplication are the same operations as for usual polynomial rings, followed by truncation of all terms of degree higher than one:

$$(x + \varepsilon.y) + (x' + \varepsilon.y') = (x + x') + \varepsilon.(y + y') \tag{2.41}$$
$$(x + \varepsilon.y) \cdot (x' + \varepsilon.y') = (x \cdot x') + \varepsilon.(x \cdot y' + x' \cdot y) \tag{2.42}$$

Orientation

The interval between cantus firmus and discantus measures their chromatic distance but tells nothing about which voice lies above the other. Furthermore, we need a notion of orientation.

Definition 12. *The **orientation** $\Omega_\pm$ is a function*

$$\begin{aligned} \Omega_\pm : \mathcal{C}_n[\varepsilon] &\longrightarrow \mathcal{C}_n \\ x + \varepsilon.y &\longmapsto x \pm y \end{aligned} \tag{2.43}$$

that projects the dual space back onto the chromatic circle.

The change of orientation is described in Sec. 29.6 of *ToM* [MGM02], where the following formula links the two sorts of orientations Ω_+ and Ω_-:

$$\Omega_- = \Omega_+ \circ ([-1]_n + \varepsilon.[2]_n) \tag{2.44}$$

Tangent space and orientation are the two ingredients necessary to define intervals in the counterpoint context.

Definition 13. *A **contrapuntal interval** ζ is a dual number. An **oriented contrapuntal interval** is a couple composed by a dual number $x + \varepsilon.y$ and an orientation Ω. The constant term x of the dual number $x + \varepsilon.y$ represents the cantus firmus, the term y of degree one, the interval. The set of all oriented contrapuntal intervals is written*

$$\hat{\mathcal{C}_n}[\varepsilon] := \mathcal{C}_n[\varepsilon] \times \{\Omega_-, \Omega_+\}. \tag{2.45}$$

The orientation is essential for real world applications and the faithful description of counterpoints. On the other hand, it is not essential for most of the theoretical investigations, and we will simplify the notation by dropping the orientation when it is not relevant, using indifferently the complete notation $\hat{\zeta} = (\zeta, \Omega)$ or only its dual number ζ to designate the same contrapuntal interval.

Given an oriented contrapuntal interval $\hat{\zeta} = (x + \varepsilon.y, \Omega)$, the cantus firmus x is directly extracted from the dual number, and the discantus needs to be retrieved by means of the orientation function Ω, as shown in (2.43). The orientation tells us if the contrapuntal interval can be interpreted as *sweeping*, i.e. where the discantus lies above $(+)$, or as *hanging*, where it lies below $(-)$ the cantus firmus. Of course, the notions of above and below have only little meaning in the circular space $\mathcal{C}_n$, but if the discantus really lies above the cantus firmus, it yields a positive orientation.

Given two midi pitches x and z of the cantus firmus and respectively the interval, we can construct an oriented contrapuntal interval $\hat{\zeta}$, if we also specify a reference pitch x_0 that specifies which is the $[0]_n$ pitch class.

$$\begin{aligned} \mathbb{Z}^2 &\longrightarrow \mathcal{C}_n^2 \times \{\Omega_-, \Omega_+\} \\ (x, z) &\longmapsto ([x - x_0]_n + \varepsilon.[|z - x|]_n, \Omega_{\mathrm{sgn}(z-x)}) \end{aligned} \tag{2.46}$$

where

$$\mathrm{sgn}\,(k) := \begin{cases} +1 & k \geq 0 \\ -1 & k < 0. \end{cases} \tag{2.47}$$

Note that, for simplicity and generality, we used integers to describe pitches without verifying if we stay inside the valid midi range that goes from 0 to 127.

The special case of the cantus firmus and the discantus having the same pitch needs a special mention. The interval class will be $[0]_n$ and the choice of orientation will not affect the resulting discantus class. The rule given in (2.47), while easy to implement, is perfectly arbitrary. One could also decide to preserve some kind of continuity in the orientation by considering the surrounding intervals.

A contrapuntal interval can be seen as an arrow with a tail anchored on the cantus firmus and a head pointing to the discantus: See Fig. 2.10.

The notion of dichotomy Δ is naturally generalised to apply in the dual space $\mathcal{C}_n[\varepsilon]$ as well.

Definition 14. *The **contrapuntal consonances** $K[\varepsilon]$ of a dual space $\mathcal{C}_n[\varepsilon]$ defined by a strong dichotomy Δ are the contrapuntal intervals whose interval belongs to the consonance set K of Δ.*

$$K[\varepsilon] := \left\{ x + \varepsilon.y \in \mathcal{C}_n[\varepsilon] \middle| y \in K \right\} \tag{2.48}$$

***Contrapuntal dissonances** $D[\varepsilon]$ are defined in the same manner, leading to the **contrapuntal dichotomy** $\Delta[\varepsilon]$ that splits the dual space into two equal parts of cardinality $n \cdot \frac{n}{2}$ each.*

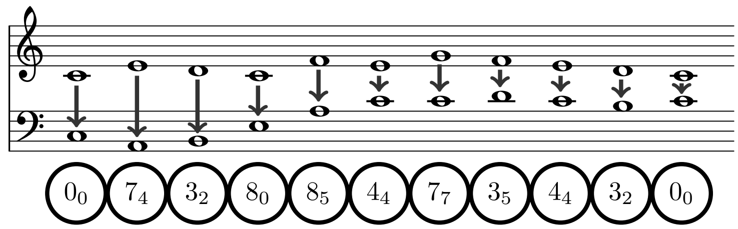


Figure 2.10: Contrapuntal intervals. The cantus firmus lies on the first staff, the discantus below in the bass staff. Arrows indicate contrapuntal intervals, which are described by the numbers below: the big number shows the distance separating the two voices, the subscript gives the position of the anchor note. Both are measured in semitones, and the numbers are residual classes modulo 12.

2.2.2 Contrapuntal Symmetries

Symmetries on the chromatic circle are affine transforms having the special property of being isomorphisms on the torus. The same holds for symmetries in the dual space: we again consider affine transforms, but this time restrict them to a subset satisfying three requirements simulating contrapuntal principles: the first one avoids repetition, the second one guarantees the compatibility with the autocomplementary function that defines the consonance-dissonance dichotomy, and the last one optimises the combinatorial possibilities of the system, which means searching the best solution for composition under the two preceding constraints. In order to do that, we first need to generalise the autocomplementary function to the dual space.

Contrapuntal Autocomplementary Functions

The autocomplementary function p_Δ of a strong dichotomy Δ has been defined for the chromatic circle in (2.31). The polarity between consonances and dissonances is extended to the dual space by means of the contrapuntal autocomplementary functions.

Definition 15. *The* ***local contrapuntal autocomplementary function*** p_Δ^x *of a strong dichotomy* Δ *at cantus firmus* x *is a bijective affine transform of the dual space*

$$p_\Delta^x : \mathcal{C}_n[\varepsilon] \longrightarrow \mathcal{C}_n[\varepsilon] \tag{2.49}$$

defined as

$$p_\Delta^x := e^{([1]_n - w)\cdot x + \varepsilon . r}.(w + \varepsilon.[0]_n) \tag{2.50}$$

where w *and* r *are the parameters of the polarity function of* Δ*:* $p_\Delta = e^r.w$*.*

The local contrapuntal autocomplementary function p_Δ^x is a *polarity* for the dichotomy $K[\varepsilon]/D[\varepsilon]$ in the sense that it permutes consonances and dissonances:

$$\begin{aligned} p_\Delta^x\left(K[\varepsilon]\right) &= D[\varepsilon] \\ p_\Delta^x\left(D[\varepsilon]\right) &= K[\varepsilon] \end{aligned} \tag{2.51}$$

The consonant interval $\varepsilon.k$ gets mapped into its dual dissonance $\varepsilon.d$ exactly as on the chromatic circle. But the cantus firmi get shuffled around the chromatic circle, excepted the contrapuntal intervals anchored on cantus firmus x

$$I_x := \left\{ x + \varepsilon.y \middle| y \in \mathcal{C}_n \right\} \tag{2.52}$$

which stay in this slice of the dual space:

$$p^x_\Delta\left(I_x\right) = I_x \tag{2.53}$$

The local contrapuntal autocomplementary function inherits all properties from the original polarity function on the chromatic circle. Its composition gives the identity.

$$p^x_\Delta \circ p^x_\Delta = Id \tag{2.54}$$

And there is also a translation formula that can be found in Sec. 30.2.2 on induced polarities in *ToM* [MGM02]:

$$p^{x+\tilde{x}}_\Delta = e^{\tilde{x}} \circ p^x_\Delta \circ e^{-\tilde{x}} \tag{2.55}$$

It is possible to define a contrapuntal autocomplementary function that preserves all cantus firmi by joining together the local autocomplementary functions of each slice I_x.

Definition 16. *The **global contrapuntal autocomplementary function** $p^\bullet_\Delta : \mathcal{C}_n[\varepsilon] \to \mathcal{C}_n[\varepsilon]$ of a strong dichotomy Δ is defined as the disjoined union of its different local contrapuntal autocomplementary functions, each one being restricted to its invariant slice of cantus firmus I_x:*

$$p^\bullet_\Delta := \biguplus_{x \in \mathcal{C}_n} p^x_\Delta|_{I_x} \tag{2.56}$$

which boils down to a much simpler expression:

$$x + \varepsilon.y \longmapsto x + \varepsilon.p_\Delta\left(y\right) \tag{2.57}$$

Note that the global autocomplementary function is not an affine transformaffine transform—) anymore.

Contrapuntal Symmetries

In the dual space, contrapuntal symmetries are the counterpart to circle symmetries. They are affine morphisms satisfying three requirements of a geometrical nature: See Chap. 31 in *ToM* [MGM02] for details on the more physical sources of inspiration of this model.

Definition 17. *A **contrapuntal symmetry** $h = e^{a+\varepsilon.b}.(u + \varepsilon.v)$ for a contrapuntal consonance $\xi = x + \varepsilon.k \in K[\varepsilon]$ of a strong dichotomy Δ is an affine bijection*

$$\begin{aligned} h : \mathcal{C}_n[\varepsilon] &\longrightarrow \mathcal{C}_n[\varepsilon] \\ \xi &\longmapsto a + \varepsilon.b + (u + \varepsilon.v) \cdot (x + \varepsilon.k) \end{aligned} \tag{2.58}$$

whereby $a, b, v \in \mathcal{C}_n$ and $u \in {\mathcal{C}_n}^\times$, have the following three properties:

1. *The contrapuntal consonance ξ is a* virtual *dissonance, i.e. its image under h is a contrapuntal dissonance:*
$$\xi \notin h\left(K[\varepsilon]\right) \tag{2.59}$$

2. *The local autocomplementary function p^x_Δ is a polarity of the transformed dichotomy:*
$$\begin{aligned} p^x_\Delta\left(h\left(K[\varepsilon]\right)\right) &= h\left(D[\varepsilon]\right) \\ p^x_\Delta\left(h\left(D[\varepsilon]\right)\right) &= h\left(K[\varepsilon]\right) \end{aligned} \tag{2.60}$$

3. *The intersection of the consonance set and its image under h must be maximal*
$$h = \operatorname{argmax}_g \left| g\left(K[\varepsilon]\right) \cap K[\varepsilon] \right| \tag{2.61}$$
among all affine bijections $g \in \mathbb{A}_n$ that satisfy the first two criteria (2.59) and (2.60).

The set of all contrapuntal symmetries of a given contrapuntal consonance ξ_0 is notated H_{ξ_0}.

The contrapuntal symmetry serves as a bridge connecting two consonances in order to define which steps, i.e. which succession of consonances, are allowed or not in a counterpoint. Tab.B.1 to B.15 lists them for chromatic gamut sizes up to $n = 12$.

Definition 18. *Let Δ be a strong dichotomy. A contrapuntal consonance $\xi_1 \in K[\varepsilon]$ is an* ***allowed successor*** *to a contrapuntal consonance $\xi_0 \in K[\varepsilon]$ if there exists (at least) one contrapuntal symmetry $h \in H_{\xi_0}$ such that*
$$\xi_1 \in \operatorname{Im}(h)\,. \tag{2.62}$$
The set of all allowed successors
$$\begin{aligned} \mathcal{C}_n[\varepsilon] &\to \mathscr{P}\left(K[\varepsilon]\right) \\ \zeta_0 &\mapsto K_{\zeta_0}[\varepsilon] \end{aligned} \tag{2.63}$$
to a given contrapuntal interval ζ_0 is defined as the union of the images of all its contrapuntal symmetries $h \in H_{\zeta_0}$.
$$K_{\zeta_0}[\varepsilon] := \begin{cases} \left(\cup_{h \in H_{\zeta_0}} h\left(K[\varepsilon]\right)\right) \cap K[\varepsilon] & \textit{if } \zeta_0 \in K[\varepsilon] \\ \varnothing & \textit{if } \zeta_0 \in D[\varepsilon] \end{cases} \tag{2.64}$$

According to (2.64), a dissonance does not have any allowed successor, nor can a dissonance follow a consonance. In counterpoint, we are always supposed to make steps from one consonance to another. The allowed steps form a strict subset of $K[\varepsilon] \times K[\varepsilon]$, as explained below.

Definition 19. *A* ***step*** *s is an ordered couple of contrapuntal intervals $(\zeta_0, \zeta_1) \in \mathcal{C}_n[\varepsilon] \times \mathcal{C}_n[\varepsilon]$. It is* ***allowed*** *with respect to a strong dichotomy Δ whenever*
$$\zeta_1 \in K_{\zeta_0}[\varepsilon] \tag{2.65}$$
holds, otherwise it is ***forbidden***.

Note that the allowed or forbidden character of a step is defined for any combination of contrapuntal intervals $(\zeta_0, \zeta_1) \in \mathcal{C}_n[\varepsilon] \times \mathcal{C}_n[\varepsilon]$.

1. If we start on a dissonance, $\zeta_0 \in D[\varepsilon]$, then $K_{\zeta_0}[\varepsilon] = \varnothing$ by (2.64) in Def. 18, so that $\zeta_1 \notin K_{\zeta_0}[\varepsilon], \forall \zeta_1 \in \mathcal{C}_n[\varepsilon]$.

$$(\zeta_0, \zeta_1) \text{ is forbidden} \quad \forall (\zeta_0, \zeta_1) \in D[\varepsilon] \times \mathcal{C}_n[\varepsilon] \tag{2.66}$$

2. If we end on a dissonance, $\zeta_1 \in D[\varepsilon]$, then $\zeta_1 \notin K_{\zeta_0}[\varepsilon], \forall \zeta_0 \in \mathcal{C}_n[\varepsilon]$ even for non empty sets of allowed successors $K_{\zeta_0}[\varepsilon]$ to a consonance $\zeta_0 \in K[\varepsilon]$, because, according to (2.64), $K_{\zeta_0}[\varepsilon] \subset K[\varepsilon]$.

$$(\zeta_0, \zeta_1) \text{ is forbidden} \quad \forall (\zeta_0, \zeta_1) \in \mathcal{C}_n[\varepsilon] \times D[\varepsilon] \tag{2.67}$$

3. The last case treats steps between two consonances $(\zeta_0, \zeta_1) \in K[\varepsilon] \times K[\varepsilon]$. The step will be allowed only if a contrapuntal symmetry $h \in H_{\zeta_0}$ relates ζ_1 to ζ_0, as defined in Def. 18.

Example 9. *In the Fuxian system, a minor third $\xi_0 = [0] + \varepsilon.[3]$ on C possesses $h = e^{[0]+\varepsilon.[8]}.([5] + \varepsilon.[4])$ as a contrapuntal symmetry. Fig. 2.11 shows how this allows one to move on further to other thirds, fifths, etc.*

Parameters

The three conditions (2.59) to (2.61) impose certain conditions on the values parameters of a contrapuntal symmetry can take. In this section we will investigate these restrictions more in detail.

First of all, we can drop the first parameter a that act as a transposition on the cantus firmus. Because of internal symmetries in the counterpoint structure, it is sufficient to consider only contrapuntal symmetries with $a = [0]_n$ for contrapuntal intervals anchored on the reference pitch class: $[0]_n + \varepsilon.k$ because the interdictions and permissions are the same, up to a transposition: $x_1 + \varepsilon.k_1$ is an allowed successor to $x_0 + \varepsilon.k_0$ iff $x_1 - x_0 + \varepsilon.k_1$ is an allowed successor to $0 + \varepsilon.k_0$. Although we will not delve into an in-depth study of this important fact, it will be used throughout this dissertation. A demonstration is given in Sec. 31.3 in *ToM* [MGM02]. We will call these simplified contrapuntal symmetries *homogeneous* and use them exclusively, but remember that there exist other (redundant) contrapuntal symmetries besides those mentioned.

The strong dichotomy Δ along with its consonance set K and its autocomplementary function $p_\Delta = e^r.w$ is given, as well as a fixed contrapuntal consonance $\xi = [0]_n + \varepsilon.k$. We want to investigate the parameters of a contrapuntal symmetry $h = e^{[0]_n+\varepsilon.b}.(u + \varepsilon.v)$. We will therefore inspect the three requirements, one after the other, in order to understand how they restrict the parameters and shape the counterpoint worlds of Sec. 3.1.

1. Virtual dissonances. Because of (2.59), ξ has to lie outside the image set of contrapuntal consonances under h, which means that there cannot exist any contrapuntal consonances $\xi_0 = x_0 + \varepsilon.k_0$ which gets mapped onto ξ.

$$h(\xi_0) = u \cdot x_0 + \varepsilon.(b + u \cdot k_0 + v \cdot x_0) \neq [0]_n + \varepsilon.k \quad \forall \xi_0 \in K[\varepsilon] \tag{2.68}$$

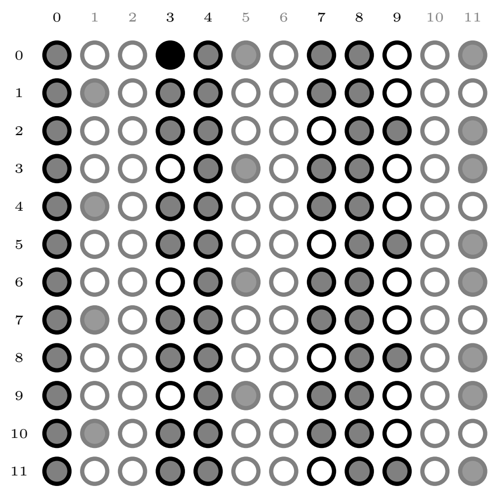


Figure 2.11: Some allowed successors to $\xi_0 = [0] + \varepsilon.[3]$. Each circle represents a contrapuntal interval. The horizontal coordinate shows the interval y, the vertical coordinate the cantus firmus x. A gray filling indicates it belongs to the image of $h = e^{[0]+\varepsilon.[8]}.([5] + \varepsilon.[4])$, a black border it is a contrapuntal consonance. The contrapuntal consonance ξ_0 itself is shown in black on the first row. Since $e^{[0]+\varepsilon.[8]}.([5] + \varepsilon.[8])$ is the second possible contrapuntal symmetry to ξ_0, its image should be added in order to get the complete set of allowed successors. Considering only this symmetry, it appears that all octaves (column $[0]$), major thirds ($[4]$) and minor sixths ($[8]$) are allowed successors, whereas there are restrictions on the other intervals.

The homogeneous character of ξ and the fact that u is a unit eliminate the cantus firmus x_0 from the above expression. We end up with a simplified condition, where the parameters u and b are determined by k and K alone:[3]

$$\boxed{k \notin b + u \cdot K}. \tag{2.69}$$

For a given u, only half of the values b could take are allowed: the transpositions that shift k outside the set $u \cdot K$.

2. Commutation with the polarity function $p_{\Delta}^{\tilde{x}} = e^{([1]_n - w)\cdot\tilde{x}+\varepsilon.r}.(w + \varepsilon.[0]_n)$.
   For each contrapuntal consonance $\xi = x + \varepsilon.k$ we get an image under $p_{\Delta}^{\tilde{x}} \circ h$

$$\begin{aligned} p_{\Delta}^{\tilde{x}} \circ h\,(x + \varepsilon.k) = \{([1]_n - w)\cdot\tilde{x} + w\cdot u\cdot x\} \\ + \varepsilon.\{r + w\cdot(b + u\cdot k + v\cdot x)\} \end{aligned} \tag{2.70}$$

   that has to be equal to the image of another consonance $\xi' = x' + \varepsilon.k'$ under the reversed composition $h \circ p_{\Delta}^{\tilde{x}}$.

$$\begin{aligned} h \circ p_{\Delta}^{\tilde{x}}\,(x' + \varepsilon.k') = \{u\cdot(([1]_n - w)\cdot\tilde{x} + w\cdot x')\} \\ + \varepsilon.\{b + u\cdot(r + wk') + v\cdot(([1]_n - w)\cdot\tilde{x} + w\cdot x')\} \end{aligned} \tag{2.71}$$

   These two equations can be simplified by dropping the offset $\tilde{x} := [0]_n$ in the autocomplementary function, since we treat the homogeneous case. Equating both parts of the dual numbers independently then yields

$$\begin{aligned} u\cdot w\cdot(x - x') = [0]_n, \\ ([1]_n - u)\cdot r + (w - [1]_n)\cdot b + u\cdot w\cdot(k - k') + v\cdot w\cdot(x - x') = [0]_n. \end{aligned} \tag{2.72}$$

   Like in point 1, he parameters u and w are both units, so the cantus firmi have to be equal.

$$x' = x \tag{2.73}$$

   The same holds for the consonances. Would it not be the case, then we should choose the parameters u and b so that the above equation were correct for the given pair of consonances $k \neq k'$. But the same equation must hold for every choice of $\tilde{k} \in K$, which in turn would imply that the non trivial translation $e^{k'-k}.[1]_n$ were an internal symmetry of the consonance set. This contradicts the fact that Δ is a *strong* dichotomy. Therefore, the consonances also cancel out and we end up with constraints on the same set of parameters as in the previous point, which are now affected by the polarity function. For a given choice of u, the interval translation parameter b has to be a solution of the following equation:[4]

$$\boxed{(w - [1]_n)\cdot b = (u - [1]_n)\cdot r.} \tag{2.74}$$

[3] See equation (31.1) on page 652 in *ToM* [MGM02].

[4] See equation (31.3) on page 652 in *ToM* [MGM02], where variable t stands for our b.

The difficulty arises from the fact that $w - [1]_n$ and $u - [1]_n$ are zero devisors. The number of possible values depends on the order of the cyclic subgroups generated by $w - [1]_n$ and $u - [1]_n$. Certain values of w may leave more freedom, hence favouring a greater amount of successors than others. See Tab. B.10. to B.14.

3. Combinatorial maximum. The choice of consonances in set K has also an impact on the last parameter v.

$$v\,(u, b, K) := \mathrm{argmax}_{\tilde{v} \in \mathcal{C}_n} \left| \{ b + \tilde{v} \cdot \mathcal{C}_n + u \cdot K \} \cap K \right| \tag{2.75}$$

There are basically three cases, depending on the kind of value v takes.

(a) $v = [0]_n$.

(b) $v \in \mathrm{ZD}\,(\mathcal{C}_n)$.

(c) $v \in {\mathcal{C}_n}^{\times}$.

In this last case, the term $v \cdot \mathcal{C}_n$ traverses all values of $\mathcal{C}_n$ and the cardinality of the intersection reaches the minimum possible value,[5] namely $\left(\frac{n}{2}\right)^2$. In the first two cases, $v \cdot \mathcal{C}_n$ restricts to a subgroup of $\mathcal{C}_n$. This can be an advantage if the set $u \cdot K$ is shifted in such a way that it is adjusted to intersect K a number of times higher than the average ($\frac{n}{2}$), and leaves out less favourable positions. In the case of $n = 12$, this is what happens.

The value of parameter v is essential in shaping the structure of counterpoint rules. For a choice of given consonances k_0 and k_1, it determines the periodicity pattern of the allowed cantus firmus x_1.

Lemma 2. *If $h = e^{[0]_n + \varepsilon . b}.(u + \varepsilon . v)$ is a contrapuntal symmetry for a contrapuntal consonance $\xi = [0]_n + \varepsilon . k$, then $\bar{h} := e^{[0]_n + \varepsilon . b}.(u - \varepsilon . v)$ is also a contrapuntal symmetry for the same consonance.*

Proof. The value of v is not affected by the first two requirements in definition 17. Only the third constraint (2.61) plays a role. So we have to prove that changing the sign of parameter v does not affect the cardinality of the intersection, i.e. that it is still maximal.

$$\left| \bar{h}\,(K[\varepsilon]) \cap K[\varepsilon] \right| = \left| \{ b + u \cdot K - v \cdot \mathcal{C}_n \} \cap K \right| \tag{2.76}$$

Since we have

$$-v \cdot \mathcal{C}_n = v \cdot (-\mathcal{C}_n) = v \cdot \mathcal{C}_n, \tag{2.77}$$

we end up with

$$\left| \bar{h}\,(K[\varepsilon]) \cap K[\varepsilon] \right| = \left| h\,(K[\varepsilon]) \cap K[\varepsilon] \right| \tag{2.78}$$

which proves the lemma. □

[5]Theorem 33 on page 653 in *ToM* [MGM02] states this result for $n = 12$. The same derivation holds for other values of the chromatic gamut size as well.

Conjecture 1. *The parameter v of a contrapuntal symmetry*

$$h = e^{a+\varepsilon.b}.u + \varepsilon.v \tag{2.79}$$

is a always a zero divisor:

$$v \in \mathrm{ZD}\,(\mathcal{C}_n) \tag{2.80}$$

This is true for values of n up to 12 (except the degenerate case of $n = 2$), see the tables in Chap. B.

2.2.3 Authorisation and Interdiction Tables

The sets of allowed successors $K_{\xi_0}[\varepsilon]$ output by the algorithm can serve to build tables showing the allowed or forbidden character of a step between contrapuntal consonances ξ_0 and ξ_1:

- authorisation

$$\xi_1 \in K_{\xi_0}[\varepsilon] \tag{2.81}$$

- interdiction

$$\xi_1 \notin K_{\xi_0}[\varepsilon] \tag{2.82}$$

Tables showing only interdictions can be made more compact, since they are far less numerous than the authorisations. The structure which appears is also more readable and the reader is not overwhelmed with uninformative signs.

Example 10. *Interdiction tables for the six strong dichotomies in $\mathcal{C}_{12}$ are shown in Tab. C.10 to C.15. Which steps are forbidden between a start contrapuntal consonance $\xi_0 = [0]_n + \varepsilon.k$ and every possible consonance $\xi_1 = x_1 + \varepsilon.k_1 n$.*

Forbidden parallels

In the interdiction tables of Chap. C, a row completely filled with crosses and having identical values for the initial and final consonances k_0 and k_1 indicates an interdiction for parallel movement of k_0-intervals. The most famous one is the fifth-parallel interdiction in the Fuxian system (Tab. C.15) but such interdictions show up in other systems as well, see the interdiction for parallels of fourths or major sevenths in strong dichotomy Δ_{64} (Tab. C.10), or major seconds or minor sixths in strong dichotomy Δ_{69}–68 in *ToM*—(Tab. C.11).

[Maz07] conjectured that there could be a connection between the unit $[7]$ that makes $K = \{[0], [3], [4], [7], [8], [9]\}$ a multplicative monoid and the interdiction of parallels of fifths. It is verified for the Fuxian dichotomy Δ_{87} in $\mathcal{C}_{12}$. But as can be seen from Tab. 2.3 and 2.4, there are cases where there are two parallel interdictions, but no multiplicative monoid, as for the DUR dichotomy Δ_{64}. The inverse also happens: the existence of multiplicative monoids but no interdiction for parallels, as for Δ_{77} (75 in *ToM*) and Δ_{78} (82 in *ToM*).

Each row shows a periodic behaviour, that is due to the parameter v of the contrapuntal symmetries, which acts systematically as a zero divisor.

Table 2.4: Forbidden parallel movements of an interval of k chromatic units. Chromatic gamut size n, strong dichotomy Δ, consonance class $[k]_n$. Strong dichotomy class numbers in $\mathcal{C}_{12}$ correspond to number 64, 68 and 82 respectively in *ToM*.

n	$[\Delta]_{\mathbb{I}_n}$	$[k]_n$
10	33	$[0]_{10}$
		$[5]_{10}$
	39	$[0]_{10}$
		$[1]_{10}$
		$[5]_{10}$
		$[6]_{10}$
	45	$[0]_{10}$
		$[3]_{10}$
		$[5]_{10}$
		$[8]_{10}$
	46	$[1]_{10}$
		$[6]_{10}$
12	64	$[5]$
		$[11]$
	69	$[0]$
		$[2]$
		$[8]$
	87	$[7]$

2.3 Summary

Counterpoint rules consist of authorisations and interdictions dictating which intervals can be used and how they can be combined. The strong dichotomies separate intervals into consonances and dissonances, and the interdiction tables tell which consonances are not allowed to follow a given consonance. Taking a strong dichotomy on input, the mathematical algorithm described in [MGM02] produces an interdiction table at the output. This procedure, originally developed for simulating the traditional Fuxian rules, also works for five other classes of strong dichotomies in the $n = 12$ context, see Fig. 2.12. In this context, the Fuxian world becomes a special case, a particular incarnation of a more general structure.

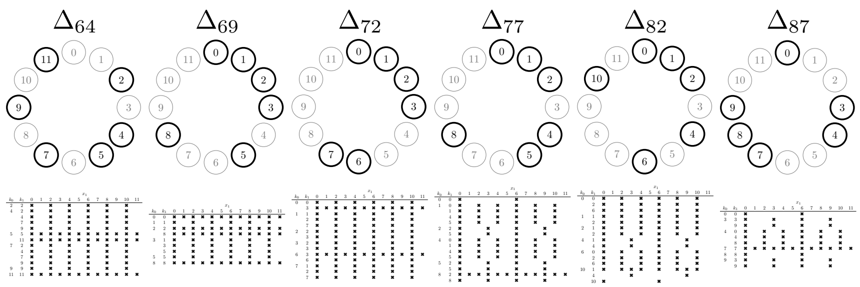


Figure 2.12: Exotic counterpoint systems in $n = 12$. Strong dichotomies are displayed on the upper row, interdiction tables on the lower one. The complete list of tables appears in Sec. C.4 at a regular size. Numbers indicate the dichotomy class $[\Delta]_{\mathbb{I}_{12}}$, and correspond respectively to 64, 68, 71, 75, 78 and 82 in *ToM*. The last one, 87, is the traditional Fuxian world. The other five classes give rise to new systems for counterpoint.

Chapter 3

The Category of Counterpoint Worlds

The previous chapter enumerated systematically all strong dichotomies and showed how each of them yields a table of forbidden steps. These two ingredients constitute a complete system of rules of composition for counterpoint, or a *counterpoint world*. They can be explored systematically, and it is the main task of this dissertation to do the same for their morphisms.

Our first contribution in Sec. 3.1 is to give a formal definition of what constitutes a system of compositional rules for counterpoint. It relies on the material presented in Chap. 2 and packs dichotomies and interdiction tables into a single object. Which kind of mappings of contrapuntal intervals is relevant for counterpoint world morphisms is discussed in Sec. 3.2. Formalising the problem setting this way yields a framework for constructing all possible morphisms, so that we will be able to fully characterise this category of counterpoint worlds.

3.1 Counterpoint Worlds

A counterpoint world is a system of rules for the composition of counterpoint mathematically generated with Mazzola's algorithm. The corpus of rules can be classified into two sorts: the first one may be called *static* and explains which intervals are allowed. The second one has a more *dynamic* flavour and decides which succession of intervals is allowed.

The former, the consonances of a strong dichotomy, form the input of the algorithm, the latter, the associated interdiction table, its output. Contrapuntal intervals are the most elementary building blocks described by the theory, and are common to all systems of rules sharing the same chromatic gamut size n. We define a *counterpoint world* by equipping the contrapuntal intervals with these two sets of counterpoint rules.

3.1.1 Global Worlds

A global counterpoint world describes how the system of compositional rules applies to the complete pool of contrapuntal consonances and dissonances, which amounts to a total of n^2 contrapuntal intervals.

Definition 20. *A* ***global counterpoint world*** $\widetilde{CW}$ *associated to a strong dichotomy* $\Delta = (K/D)$ *is a triple* $(\tilde{\kappa}, \tilde{\sigma}, \widetilde{p^{\bullet}_{\Delta}})$ *whose components are*

1. *A consonance indicator function* $\tilde{\kappa}$ *which tells if a contrapuntal interval is consonant or not:*

$$\begin{aligned} \tilde{\kappa} : \mathcal{C}_n[\varepsilon] &\longrightarrow \{\top, \bot\} \\ \zeta &\longmapsto \begin{cases} \top & \textit{if } \zeta \in K[\varepsilon], \\ \bot & \textit{otherwise.} \end{cases} \end{aligned} \tag{3.1}$$

2. *An allowed step indicator function* $\tilde{\sigma}$ *which tells if a succession of two contrapuntal intervals is allowed or not:*

$$\begin{aligned} \tilde{\sigma} : \mathcal{C}_n[\varepsilon] \times \mathcal{C}_n[\varepsilon] &\longrightarrow \{\top, \bot\} \\ (\zeta_0, \zeta_1) &\longmapsto \begin{cases} \top & \textit{if } \zeta_1 \in K_{\zeta_0}[\varepsilon] \\ \bot & \textit{otherwise.} \end{cases} \end{aligned} \tag{3.2}$$

3. *The associated global autocomplementary function* $\widetilde{p^{\bullet}_{\Delta}}$ *which maps contrapuntal consonances into dissonances and vice-versa:*

$$\begin{aligned} &\widetilde{p^{\bullet}_{\Delta}} : \mathcal{C}_n[\varepsilon] \longrightarrow \mathcal{C}_n[\varepsilon] \\ &\widetilde{p^{\bullet}_{\Delta}}\left(K[\varepsilon]\right) = D[\varepsilon] \\ &\widetilde{p^{\bullet}_{\Delta}}\left(D[\varepsilon]\right) = K[\varepsilon] \end{aligned} \tag{3.3}$$

The set of contrapuntal consonances $K[\varepsilon]$ was defined in Def. 14, and the set of allowed successors $K_{\zeta_0}[\varepsilon]$ to a contrapuntal interval ζ_0 in Def. 18. How it can serve to define allowed and forbidden steps is the topic of Def. 19 and the discussion following it.

The functions $\tilde{\kappa}$ and $\tilde{\sigma}$ express the compositional rules. Given a global counterpoint world, the contrapuntal consonances could be retrieved by

$$K[\varepsilon] = \tilde{\kappa}^{-1}\left(\top\right) \tag{3.4}$$

as well as the set of allowed steps:

$$\tilde{\sigma}^{-1}\left(\top\right). \tag{3.5}$$

The contribution of the third element, the polarity function $\widetilde{p^{\bullet}_{\Delta}}$, is more structural. It acts like a mirror on the set of contrapuntal intervals, building pairs between the prohibited world of dissonances and the legal one of consonances. These three elements taken together constitute the essential structure of a counterpoint world and carry the musical meaning attached to contrapuntal intervals.

3.1.2 Local Worlds

A piece of music, or counterpoint, often uses only a dozen consonances (and sometimes dissonances), rarely the total set of 144 contrapuntal intervals (when $n = 12$). Caring about the unused intervals when transforming a counterpoint is an unnecessary burden and—of utmost importance—may compromise the task. As we will see in Sec. 5.4, some counterpoint worlds simply do not fit into any other.

In such cases, the only way to proceed is to restrict and reduce the problem to the minimal set of necessary intervals, defined next.

Definition 21. *The subset of contrapuntal intervals $\mathcal{R}$* ***relevant*** *for a given (arbitrary) set of contrapuntal intervals $\mathcal{Z}$ in a counterpoint world $\widetilde{CW} = (\tilde{\kappa}, \tilde{\sigma}, \widetilde{p^{\bullet}_{\Delta}})$ consists of all its members along with their duals.*

$$\mathcal{R} := \mathcal{Z} \cup \widetilde{p^{\bullet}_{\Delta}}\,(\mathcal{Z}) \tag{3.6}$$

Note that this is not a disjoint union: $\mathcal{Z}$ may contain consonances as well as dissonances, and possibly both members of a dual pair, so that the original and dual sets may perfectly overlap: $\left|\mathcal{Z}\right| \leq \left|\mathcal{R}\right| \leq 2\cdot\left|\mathcal{Z}\right|$. The next example illustrates this phenomenon.

Example 11. *Let Δ be the traditional Fuxian consonances along with their polarity function $p_{\Delta} = e^{[2]}.[5]$, whose action on consonances and dissonances is shown in Fig. 2.8. Let $\mathcal{Z}$ be the set of following four contrapuntal intervals:*

$$\mathcal{Z} := \{[0] + \varepsilon.[1], [1] + \varepsilon.[0], [1] + \varepsilon.[2], [2] + \varepsilon.[8]\}$$

Its dual counterpart becomes

$$\widetilde{p^{\bullet}_{\Delta}}\,(\mathcal{Z}) := \{[0] + \varepsilon.[7], [1] + \varepsilon.[2], [1] + \varepsilon.[0], [2] + \varepsilon.[6]\},$$

yielding a relevant set of six contrapuntal consonances and dissonances.

$$\mathcal{R} = \{[0] + \varepsilon.[1], [0] + \varepsilon.[7], [1] + \varepsilon.[0], [1] + \varepsilon.[2], [2] + \varepsilon.[6], [2] + \varepsilon.[8]\}$$

A local counterpoint world is nothing else than a global counterpoint world restricted to a set of relevant contrapuntal intervals.

Definition 22. *A* ***local counterpoint world*** *CW of a global counterpoint world $\widetilde{CW} = (\tilde{\kappa}, \tilde{\sigma}, \widetilde{p^{\bullet}_{\Delta}})$ induced by the contrapuntal interval set $\mathcal{Z} \subseteq \mathcal{C}_n[\varepsilon]$ is a couple $(\widetilde{CW}, \mathcal{R})$, where $\mathcal{R}$ is the set of relevant contrapuntal intervals induced by $\mathcal{Z}$. The local indicator and polarity functions $(\kappa, \sigma, p^{\bullet}_{\Delta})$ can be naturally defined as restrictions of the global functions on the set of relevant contrapuntal intervals:*

1. *The local consonance indicator function κ tells which relevant contrapuntal interval in $\mathcal{R}$ is consonant:*

$$\begin{aligned} \kappa : \mathcal{R} &\longrightarrow \{\top, \bot\} \\ \zeta &\mapsto \tilde{\kappa}|_{\mathcal{R}}\,(\zeta)\,. \end{aligned} \tag{3.7}$$

2. *The local allowed step indicator function σ tells whether a succession of two relevant contrapuntal intervals is allowed or not:*

$$\begin{aligned} \sigma : \mathcal{R} \times \mathcal{R} &\longrightarrow \{\top, \bot\} \\ (\zeta_0, \zeta_1) &\longmapsto \tilde{\sigma}\big|_{\mathcal{R}\times\mathcal{R}} (\zeta_0, \zeta_1)\,. \end{aligned} \tag{3.8}$$

3. *The local global autocomplementary function $p^\bullet_\Delta$ maps relevant contrapuntal consonances into their dual dissonances and vice-versa:*

$$\begin{aligned} p^\bullet_\Delta : \mathcal{R} &\longrightarrow \mathcal{R} \\ \zeta &\longmapsto \widetilde{p^\bullet_\Delta}\Big|_{\mathcal{R}} (\zeta)\,. \end{aligned} \tag{3.9}$$

Technically speaking, the structure of a local counterpoint world does not differ from a global counterpoint world. Taking a trivial example, the local counterpoint world CW induced by the whole set of contrapuntal intervals $\mathcal{Z} = \mathcal{C}_n[\varepsilon]$ is the global world $\widetilde{CW}$ itself.

All the following theoretical considerations about morphisms and graphs in Chap. 4 and 5 apply to both. The only difference appears at usage, where morphisms certainly do not exist for certain global counterpoint worlds, but may exist for some of their local worlds.

For the sake of simplicity we will unify the notation so that a counterpoint world CW, without further specification, will always designate a local counterpoint world restricted to some set of relevant contrapuntal intervals $\mathcal{R}$ (that may possibly be the complete dual space). It will be denoted by the triple of local functions $(\kappa, \sigma, p^\bullet_\Delta)$ instead of the more precise couple $(\widetilde{CW}, \mathcal{R})$. The relevant intervals can be retrieved by means of the local function's domains: $\mathcal{R} = \mathrm{Dom}\,(\kappa)$ or $\mathcal{R} = \mathrm{Dom}\,(p^\bullet_\Delta)$.

3.1.3 Counterpoints

We give here a formal definition of a counterpoint, meaning a piece of music resulting from the application of compositional rules.

Definition 23. *A **counterpoint** p in $\mathcal{C}_n$ is a finite ordered sequence of $S+1$ oriented contrapuntal intervals.*

$$\begin{aligned} p : \{0, \ldots, S\} &\longrightarrow \hat{\mathcal{C}_n}[\varepsilon] \\ s &\longmapsto \hat{\zeta}_s = (x_s + \varepsilon.y_s, \Omega_s) \end{aligned} \tag{3.10}$$

When orientation is not relevant, we simply consider a sequence of contrapuntal intervals $(\zeta_s)_{s=0}^S$.

The set of contrapuntal intervals appearing in a counterpoint p is given by its image set $\mathcal{Z} = \mathrm{Im}\,(p)$. A local counterpoint world induced by a counterpoint is the restriction of a global counterpoint world to the set of contrapuntal intervals relevant to the counterpoint.

3.2 Counterpoint World Morphisms

For a mapping between counterpoint worlds to make sense, the additional structure that distinguishes a counterpoint world from an ordinary set of contrapuntal intervals must be preserved. It does not matter if the result sounds completely weird compared to the original. Even if we end up mapping pitch classes and interval classes, the operation is not so much about acting on these objects themselves, than about mapping the rules that govern their usage.

Def. 20 states a triple, so there are three aspects we must care about when transforming a counterpoint p, which is a good citizen in a given counterpoint world $CW = (\kappa, \sigma, p^{\bullet}_{\Delta})$ into a new composition p' that behaves in accordance with the rules applying in another counterpoint world $CW' = (\kappa', \sigma', p^{\bullet}_{\Delta'})$: the character of a single contrapuntal interval (is it a consonance or not?), the character of a single step (is it allowed or not?) and the dual pairings between consonances and dissonances. A counterpoint world morphism is basically a mapping of a relevant subset $\mathcal{R}$ of the contrapuntal intervals $\mathcal{C}_n[\varepsilon]$ (possibly the complete set in case of a global world), that should ensure compatibility on these three levels.

Definition 24. *A* ***counterpoint world morphism*** ψ *between two counterpoint worlds $CW = (\kappa, \sigma, p^{\bullet}_{\Delta})$ and $CW' = (\kappa', \sigma', p^{\bullet}_{\Delta'})$, is a set function*

$$\psi : \mathcal{R} \longrightarrow \mathcal{R}' \tag{3.11}$$

satisfying the three following requirements:

1. *Preservation of the consonance structure*

$$\kappa(\zeta) = \kappa' \circ \psi(\zeta) \quad \forall \zeta \in \mathcal{R} \tag{3.12}$$

Which is equivalent to the commutativity of the following diagram in the category of sets $\mathfrak{Set}$.

$$\begin{tikzcd} \mathcal{R} \arrow[rr, "\psi"] \arrow[dr, "\kappa"'] & & \mathcal{R}' \arrow[dl, "\kappa'"] \\ & \{\top, \bot\} & \end{tikzcd}$$

2. *Compatibility with the step structure*

$$\sigma(\zeta_0, \zeta_1) = \sigma'(\psi(\zeta_0), \psi(\zeta_1)) \quad \forall \zeta_0, \zeta_1 \in \mathcal{R} \tag{3.13}$$

$$\begin{tikzcd} \mathcal{R} \times \mathcal{R} \arrow[rr, "\psi \times \psi"] \arrow[dr, "\sigma"'] & & \mathcal{R}' \times \mathcal{R}' \arrow[dl, "\sigma'"] \\ & \{\top, \bot\} & \end{tikzcd}$$

3. *Compatibility with the polarity structure*

$$p^{\bullet}_{\Delta'} \circ \psi\left(\zeta\right) = \psi \circ p^{\bullet}_{\Delta}\left(\zeta\right) \quad \forall \zeta \in \mathcal{R} \tag{3.14}$$

$$\begin{tikzcd} \mathcal{R} \cap K[\varepsilon] \arrow[r, tail, two heads, "p^{\bullet}_{\Delta}"] \arrow[d, "\psi"'] & D[\varepsilon] \cap \mathcal{R} \arrow[d, "\psi"] \\ \mathcal{R}' \cap K'[\varepsilon] \arrow[r, tail, two heads, "p^{\bullet}_{\Delta'}"'] & D'[\varepsilon] \cap \mathcal{R}' \end{tikzcd}$$

In definition 21, relevant contrapuntal intervals are constructed to include both contrapuntal consonances and their dual dissonances, so that the consonance indicator functions κ and κ' are always surjective. This is not always true for the step indicator functions σ and σ'. One could perfectly imagine a counterpoint following strictly allowed steps, or the opposite case, only forbidden ones.

Note that for a morphism to exist, it is not necessary that both worlds CW and CW' be defined on the same set of contrapuntal intervals $\mathcal{C}_n[\varepsilon]$, i.e. share the same chromatic gamut size n. The requirements concern only the logical character of interdictions and authorisations, not the algebraic nature of contrapuntal intervals. It thus becomes possible to transform a Fuxian counterpoint into a macro- or micro-tonal piece of music.

Total Symmetry

Def. 24 requires more than the minimal conditions ensuring a valid transformation of a counterpoint. One could just take care of preserving the legal character of intervals and steps, ignoring the forbidden side. Equations (3.12) and (3.13) would then become:

$$\begin{aligned} \kappa\left(\zeta\right) = \top \Rightarrow \kappa' \circ \psi\left(\zeta\right) = \top \quad \forall \zeta \in \mathcal{C}_n[\varepsilon], \\ \sigma\left(\zeta_0, \zeta_1\right) = \top \Rightarrow \sigma'\left(\psi\left(\zeta_0\right), \psi\left(\zeta_1\right)\right) = \top \quad \forall \zeta_0, \zeta_1 \in \mathcal{C}_n[\varepsilon]. \end{aligned} \tag{3.15}$$

In other words, nothing prevents a dissonance from being mapped into a consonance, and a forbidden step into an allowed one. Not to mention that we would completely ignore the polarity functions and the duality pairing of consonances and dissonances.

This lighter procedure produces a perfectly valid counterpoint. So why should we restrict the possibilities of building a counterpoint transformation? There are two reasons, the first one has to deal with theory, the second one is of practical concern.

The three conditions of Def. 24 imply a perfect symmetry between allowed and forbidden intervals and steps. Its direct benefit is a reduction of the dimension and size of the problem. Sec. 4.1 will explain this in detail and show how mapping only a part of the intervals, not the whole, is sufficient. The drawback of this trick is that we have to make sure that the rest of the intervals get treated properly, i.e. that no allowed steps get mapped into a forbidden one, according to the minimal requirements (3.15) that cannot be violated.

The practical aspect is that we are able to treat any counterpoint in any counterpoint world, not only good citizens composed for a particular counterpoint world. A real counterpoint may sometimes violate the rules, especially if we embed the same counterpoint in different counterpoint worlds in an unrestricted way. Our totally symmetric approach will map any deviation faithfully.

3.3 Summary

The counterpoint rules were split into two distinct parts: strong dichotomies and interdiction tables. The concept of a counterpoint world packs these two objects into a single definition.

For every counterpoint world CW, the identity set function $Id : \mathcal{C}_n[\varepsilon] \longrightarrow \mathcal{C}_n[\varepsilon]$ preserves the contrapuntal structure and is thus a counterpoint world morphism. One can easily verify, by concatenating the previous commutative diagrams, that the structures are also preserved by function composition inherited from $\mathfrak{Set}$. All requirements are met for these objects and morphisms to build the category of counterpoint worlds and counterpoint world morphisms $\mathfrak{Ctp}$.

Chapter 4

Graphs

Besides some minor discrepancies, the main difference between the mathematically generated authorisation/interdiction tables and the empirically developed Fuxian rules lies on a structural level. While there are a few cases in traditional counterpoint where three successive intervals must be considered, e.g. when attacking fifths and octaves with contrary movement, see [Tit59], the model always involves two consecutive contrapuntal intervals (in this sense it is analogous to a first order Markov process).

This first order nature of the model naturally yields a graph representation. It induces a binary relation on the set of all contrapuntal intervals $\mathcal{C}_n[\varepsilon]$, which in turn defines the adjacency matrix of a directed graph (or *digraph*) D.

The most straightforward representation of a counterpoint world by means of a graph would be a directed graph whose vertices consist of all or only the necessary subset of the n^2 contrapuntal intervals, and whose arrows represent allowed steps. A graph homomorphism[1] then automatically expresses the idea of preserving the validity of a counterpoint: an allowed step would always be mapped into an allowed one, exactly the way stated in the minimal requirements (3.15).

In this chapter, it will appear clearly how, on the theoretical level, the more restrictive point of view adopted in Sec. 3.2 yields a much cleaner graph implementation. Consequences on the algorithmic level imply the size reduction of the data, a better organisation, and a dramatic reduction of the search tree. We thus won't use the complete category $\mathfrak{DGph}$ of directed graphs and their homomorphisms, but a subcategory, tailored to the special task of constructing counterpoint world morphisms.

This subcategory of strict digraphs is defined in Sec. 4.1. Sec. 4.2 shows how they can be simplified with the help of quotient graphs, and Sec. 4.3 describes their inclusion hierarchy. A similar procedure follows for the morphisms in Sec. 4.3.2, so that quotient graphs will be extensively used in the algorithm presented in Chap. 5 for enumerating counterpoint world morphisms. Sec. 4.4 illustrates how the graph theoretical formalism developed in this chapter can be applied to investigate the compatibility de-

[1] A mapping $\phi : D \to D'$ of the vertices that preserves the arrows, see volume 1 of [MMW04]:

$$(v_0, v_1) \in A(D) \Rightarrow (\phi(v_0), \phi(v_1)) \in A(D') \quad \forall v_0, v_1 \in V(D). \tag{4.1}$$

gree of counterpoint rules with musical scales, using simple graph parameters such as the number of vertices and arrows.

4.1 The Category of Strict Digraphs

We define our working category as a subcategory of the more general category of digraphs and graph homomorphisms $\mathfrak{DGph}$, by requiring additional properties from objects and morphisms.

Definition 25. *A* ***strict digraph*** $D = (V, A)$ *is a simple and reflexive directed graph.*

$$(v, v) \in A \quad \forall v \in V \tag{4.2}$$

There are no multiple arrows, i.e. no more than one arrow connecting two vertices, but a loop is attached to every vertex.

Definition 26. *A* ***strict digraph morphism*** $\phi : D \to D'$ *from a strict digraph* $D = (V, A)$ *to a strict digraph* $D' = (V', A')$ *is a directed graph homomorphism that not only preserves arrows, but also the absence of arrows.*

$$(v_0, v_1) \in A \Leftrightarrow (\phi(v_0), \phi(v_1)) \in A' \quad \forall v_0, v_1 \in V \tag{4.3}$$

Identity homomorphisms preserve both absence and presence of arrows. Composition of general homomorphisms preserve the presence of arrows, and also the absence if they are strict. Strict digraph morphisms inherit from the enclosing category of digraphs $\mathfrak{DGph}$ the correct behaviour of morphism composition. We define $\mathfrak{SDGph}$ as the category of strict digraphs and strict digraph homomorphisms, a subcategory of $\mathfrak{DGph}$.

Association of a strict digraph with a counterpoint world occurs by means of the strict digraph functor, whose operation is illustrated in Fig. 4.1.

Definition 27. *The* ***strict digraph functor***

$$\begin{aligned} \mathcal{D} : \mathfrak{Ctp} &\longrightarrow \mathfrak{SDGph} \\ CW &\mapsto D \\ \psi &\longmapsto \phi \end{aligned} \tag{4.4}$$

associates a strict digraph $D = (V, A)$ *to a a counterpoint world* $CW = (\kappa, \sigma, p^{\bullet}_{\Delta})$ *in the following way:*

1. *The vertex set* V *is the set of all relevant contrapuntal consonances.*

$$V := \left\{ \zeta \in \mathcal{R} \middle| \kappa(\zeta) = \top \right\} \tag{4.5}$$

2. *The arrow set* A *represents all forbidden steps between relevant contrapuntal consonances.*

$$A := \left\{ (\zeta_0, \zeta_1) \in V \times V \middle| \sigma(\zeta_0, \zeta_1) = \bot \right\} \tag{4.6}$$

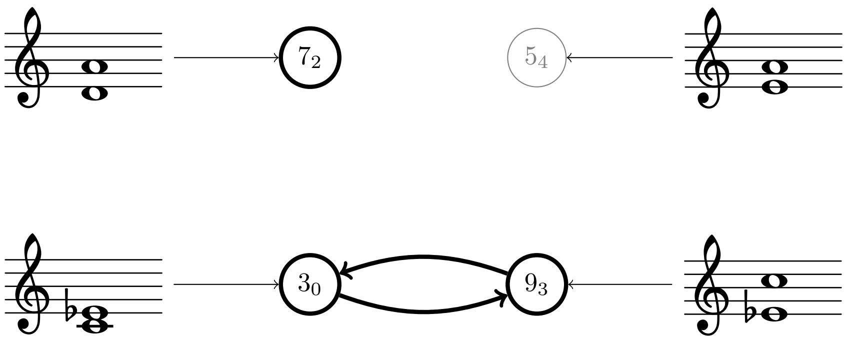


Figure 4.1: The strict digraph functor $\mathcal{D}$ associates contrapuntal consonances to vertices: e.g. in the Fuxian world Δ_{87}, a fifth (7 semitones) anchored on a D (2 semitones above C). But there are no vertices that serve as images for contrapuntal dissonances, like a fourth (5) anchored on an E (4). Forbidden steps between consonances yield arrows, here the bidirectional interdiction between a minor third (3) on a C (0) and a major (9) on an $E\flat$ (3). For readability, loops are not displayed. The designation of vertices follows the notation for quotient components introduced in (4.16).

3. *A strict digraph morphism ϕ is the restriction of a counterpoint world morphism $\psi : CW \to CW'$ to the relevant contrapuntal consonances.*

$$\begin{aligned} \phi : D &\longrightarrow D' \\ \xi &\longmapsto \psi|_V(\xi) \end{aligned} \tag{4.7}$$

The essential difference between a counterpoint world morphism ψ and its induced strict digraph morphism ϕ is that the former maps all relevant contrapuntal intervals (including dissonances), while the latter is defined on the consonances alone.

The association between counterpoint world and strict digraph morphisms is one-to-one. On the vertex level, it is not possible for two different counterpoint world morphisms ψ_1 and ψ_2 to induce an identical strict digraph morphisms ϕ, nor is it possible for a single ψ to induce two different ϕ. This is due to definition (4.7), where both counterpoint world morphisms have to agree on the set of consonances.

$$\psi_1|_V(\xi) = \phi(\xi) = \psi_2|_V(\xi) \quad \forall \xi \in V \tag{4.8}$$

And the compatibility with the polarity structure (3.14) imposes an identical mapping of the dual set of consonances. Dissonances are handled by means of the polarity function.

$$\begin{aligned} \psi_1(\eta) = \psi_1 \circ p^{\bullet}_{\Delta}(\xi) &= p^{\bullet}_{\Delta'} \circ \psi_1(\xi) \\ &= p^{\bullet}_{\Delta'} \circ \psi_2(\xi) = \psi_2 \circ p^{\bullet}_{\Delta}(\xi) = \psi_2(\eta) \quad \forall \eta \in \mathcal{R} \backslash V \end{aligned} \tag{4.9}$$

What happens on the arrow level depends on how we define forbidden steps. Remember that the step indicator functions were defined in (3.2) to evaluate to true only if both involved intervals are consonances. Steps between two dissonances or mixed steps between consonances and dissonances will always be forbidden, as explained in Def. 19. At the first sight, this may contradict the efforts made in Sec. 3.2 in treating consonances and dissonances in a perfectly symmetrical way. We could have defined allowed steps between two dissonances η_0 and η_1 if they both were duals of consonances connected by an allowed step:

$$\sigma(\eta_0, \eta_1) := \sigma\left(p^{\bullet}_{\Delta}(\eta_0), p^{\bullet}_{\Delta}(\eta_1)\right) \quad \forall \eta_0, \eta_1 \in D[\varepsilon]. \tag{4.10}$$

But steps mixing different characters of contrapuntal intervals should be prohibited anyway, since they are not in the spirit of the first point in Def. 17: how could we define virtual consonances or dissonances in this case? Nevertheless, such extensions to the theory are not relevant, since they deal with cases that are excluded from the Fuxian corpus of rules. Forbidding steps between contrapuntal intervals as soon as a dissonance is involved (as a start or end interval) also facilitates the reconstruction of a complete counterpoint world morphism even if only steps between consonances are known, as it will appear in Sec. 5.3, thus ensuring a one-to-one relationshsip between strict digraphs and counterpoint world morphisms.

$$\mathcal{D}^{-1}(\zeta_0, \zeta_1) := \begin{cases} \top & \kappa(\zeta_0) = \top \wedge \kappa(\zeta_1) = \top \wedge (\zeta_0, \zeta_1) \in A(D) \\ \bot & \text{otherwise} \end{cases} \tag{4.11}$$

Any case not treated by the strict digraph is by definition a forbidden step. And since strict digraphs do not have multiple arrows, this reverse assignment is unique and well defined.

In (4.5) we incorporate only half of all contrapuntal intervals into the vertex set, so that the resulting graphs benefit from a reduction of 50% in their order, and about the same for their size: shrinking from 75% down to 25% the amount of all arrows a digraph would contain, as shown in Tab. 4.1. Most software implementations use an incidence list for representing graphs, which is particularly effective for such sparse graphs, see [Wes01] and [Ski98]. Less storage is needed, and algorithms perform faster. This is a technical argument in favour of the requirements for total symmetry defined in Sec. 3.2.

The choice of consonances may be musically relevant, but is arbitrary on the mathematical level: the polarity function $p^{\bullet}_{\Delta}$ ensures a perfect symmetry and a one-to-one correspondence between consonances and dissonances (if we leave aside the aforementioned question of dual steps between dissonances that could theoretically be allowed), making the second part redundant. On the contrary, taking either authorised or forbidden steps has consequences on the computational level. Forbidden steps as defined in (4.6) should be favoured because they are far less numerous than the allowed ones—were it not the case, it would be impossible to compose music in such an over-restrictive system! Information always lies in variations, irregularities and exceptions, in this case the forbidden steps. Fig. 4.2 illustrates this principle.

A last word about the nature of the strict digraphs connection structure, seen as a binary relation. The reflexive character of a strict digraph is necessary: there is an interdiction loop attached to every vertex. This is due to the first property of a

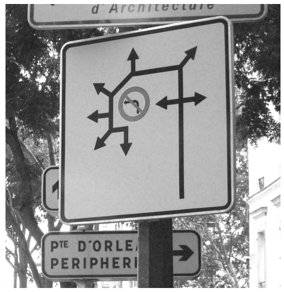


Figure 4.2: A strict digraph associated to a counterpoint world works exactly like traffic signs do: usually, only the single (or at least a few) forbidden routes is mentioned. One can see how complicated the representation would become if all remaining allowed possibilities were displayed instead.

contrapuntal symmetry (2.59) which imposes variation and thus prohibits repetition of the same contrapuntal interval. There is however no symmetry and no transitivity in general, as can be seen from Fig. 4.3. However, it will appear later in Sec. 4.2 that these properties hold for entire blocks, or subgraphs.

Simple graph parameters have a straightforward contrapuntal interpretation. The number of forbidden steps that cannot—and should not—be made from a given contrapuntal consonance ξ_0 is read directly from the strict digraph: it is the out-degree $a^{(1)}$, or number of (forbidden) successors. Tab. 4.2 shows their distribution for the six common counterpoint worlds.

The assignment of worlds to strict digraphs is injective. The functor is fully-faithful, i.e., $\mathcal{D}$ is an isomorphism of $\mathfrak{Ctp}$ onto a full subcategory of strict digraphs. It is equivalent to explore all possible counterpoint world morphisms of $\mathfrak{Ctp}$ among their corresponding strict digraph morphisms in $\mathfrak{SDGph}$.

4.1.1 Counterpoint Worlds and Counterpoints

The graph theoretical flavour of the objects defined in the previous chapter are further brought into light by the strict digraph functor. A global counterpoint world, a local counterpoint world and a counterpoint are naturally translated into their graph theoretical pendants: a strict digraph, an induced strict sub digraph, a directed walk, respectively. Fig. 4.3 illustrates this process for global counterpoint worlds.

According to definition 23, a counterpoint $p : \{0, \ldots, S\} \to \mathcal{C}[\varepsilon]$ is a succession of $S+1$ contrapuntal intervals (ignoring orientation in this case).

$$\zeta_0 \xrightarrow{a_1} \zeta_1 \xrightarrow{a_2} \cdots \xrightarrow{a_{S-1}} \zeta_{S-1} \xrightarrow{a_S} \zeta_S \tag{4.12}$$

Nothing prevents the same contrapuntal interval ζ, as well as the same step a, or pair of successive contrapuntal intervals, from appearing several times in the sequence. Only

Table 4.1: Parameters of the strict digraphs. Chromatic gamut size n, strong dichotomy class number $[\Delta]_{\mathbb{I}_n}$, number of vertices in the digraph $n(D)$, average count of forbidden successors $a(D)/n$ and proportion of forbidden steps $a(D)/n^2$. Note how the Fuxian world 87 is the least restrictive of all. Strong dichotomy class numbers in $\mathcal{C}_{12}$ correspond to 64, 68, 71, 75, 78 respectively 82 in *ToM*.

n	$[\Delta]_{\mathbb{I}_n}$	$n(D)$	$a(D)/n$	$a(D)/n^2$
2	2	2	1	50.0%
6	7	18	11	20.4%
8	13	32	32	25.0%
	16		32	25.0%
10	33	50	75	30.0%
	36		51	20.4%
	39		85	34.0%
	44		51	20.4%
	45		85	34.0%
	46		75	30.0%
12	64	72	96	22.2%
	69		78	18.1%
	72		96	22.2%
	77		82	19.0%
	82		80	18.5%
	87		50	11.6%

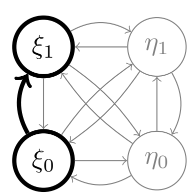
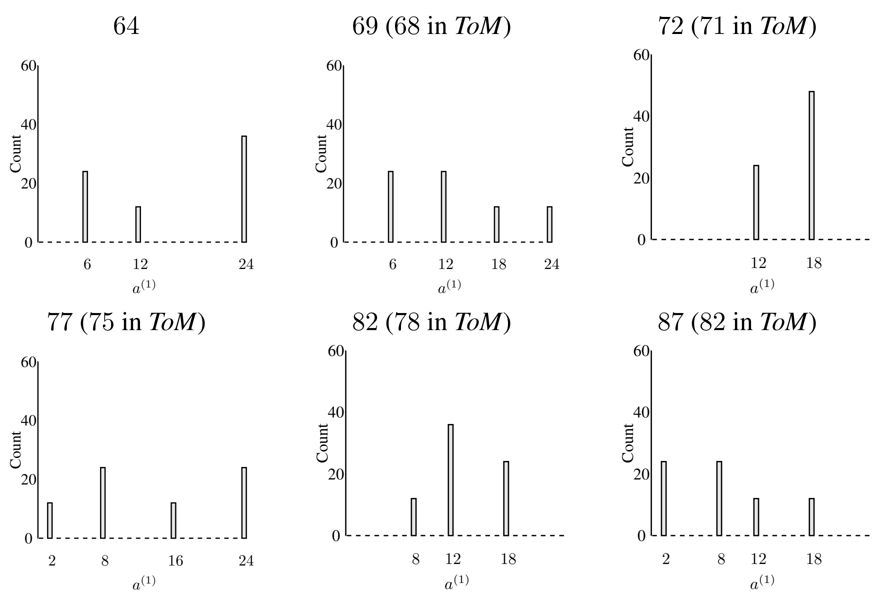


Figure 4.3: A sample of the constitutive parts of a global strict digraph D. Elements displayed with thick lines are part of the strict digraph: consonances ξ_0 and ξ_1, as well as the forbidden step connecting them. Dissonances η_0 and η_1 and allowed steps, not part of the strict digraph, are displayed in gray. Loops have been omitted for readability, but the strict digraph would also contain the two attached to ξ_0 and ξ_1.

Table 4.2: Distribution of out-degrees $a^{(1)}$ of strict digraphs induced by the six dichotomy classes of $\mathcal{C}_{12}$. Bar height indicates the number of contrapuntal consonnaces with a given out-degree. Less restrictive intervals, i.e. allowing more successors, are situated on the left side of the horizontal axis, more restrictive intervals on the right side.



direct repetitions of a same interval are forbidden. This is the reason why we have to consider the more general case of a *directed walk*, and can't use a *trail* nor a *path*.[2]

If a counterpoint follows the rules of a particular counterpoint world CW, it will be a walk in the complement[3] of its associated strict digraph $D = \mathcal{D}(CW)$, as shown in Fig. 4.4.

$$\begin{aligned} p(s) &\in V(D) \quad \forall s \in \{0, \ldots, S\} \\ (p(s-1), p(s)) &\notin A(D) \quad \forall s \in \{1, \ldots, S\} \end{aligned} \tag{4.13}$$

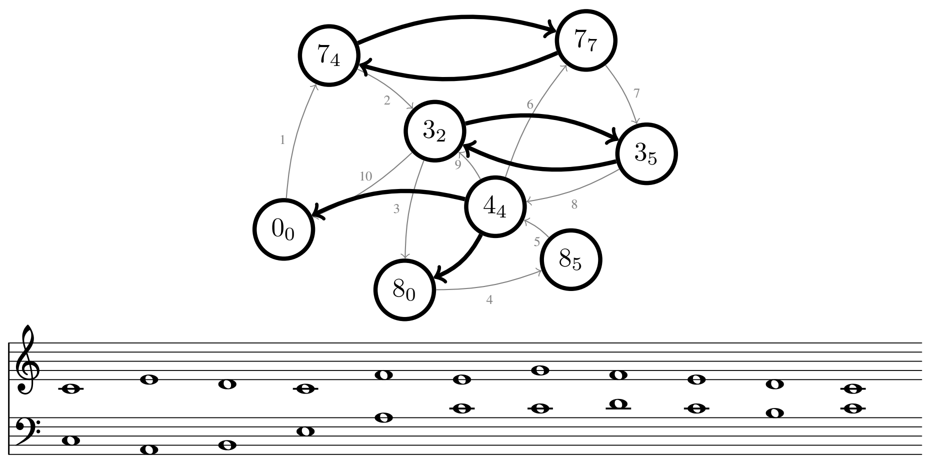


Figure 4.4: A counterpoint p following the Fuxian rules of world Δ_{87}. Each step s_i (gray arrows) carefully avoids interdictions (bold arrows), yielding a walk in the complementary digraph D^c of authorisations. For readability, all forbidden loops have been omitted. The upper staff contains the cantus firmus x_i, whose positions appear in the subscripts. The distance to the discantus k_i in the lower staff is shown by the main numbers. Both are residual classes modulo 12.

Would it use only forbidden steps, it would be a walk in the strict digraph itself. And would it use a mixture of both, it would be a walk in the complete digraph $D \cup D^c$, where D^c designates the complementary graph, see Fig. 4.5.

Notice that definition 23 does not mention any particular counterpoint world. This gives maximum freedom for transforming any counterpoint between arbitrary counterpoint worlds.

[2]The distinction between these three terms is made in definition 1.2.2 on page 20 in [Wes01], and also applies for the directed case, as mentioned on page 60. This differs from the definition given in [MMW04], where the terms *path*—definition 69, page 107—and *walk*— definition 74 page 112—serve to distinguish between the directed and undirected cases respectively, both designing what West calls a path. A counterpoint is thus a path in Mazzola's nomenclature, or a directed walk according to West's definitions.

[3]The complementary digraph D^c to a digraph D has a complementary arrow set:

$$\begin{aligned} V(D^c) &:= V(D) \\ A(D^c) &:= \left\{ (v_0, v_1) \in V \times V \middle| (v_0, v_1) \notin A(D) \right\} \end{aligned}$$

Local counterpoint worlds become subgraphs induced by the relevant consonances of a particular counterpoint. Fig. 4.5 illustrates this process.

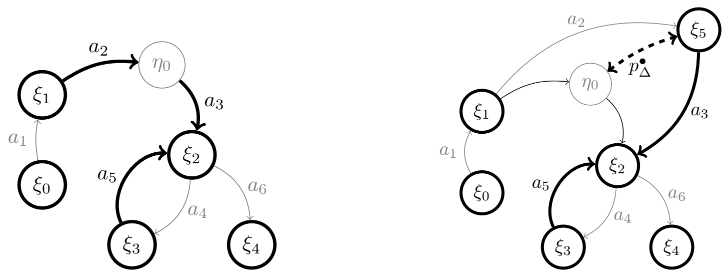


Figure 4.5: On the left, a counterpoint $p = \xi_0 - a_1 - \xi_1 - a_2 - \eta_0 - a_3 - \xi_2 - a_4 - \xi_3 - a_5 - \xi_2 - a_6 - \xi_4$. It may use consonances (ξ_i) as well as dissononaces (η_0), and walk along forbidden steps a_2, a_3, and a_5, and allowed ones, passing more than once through vertex ξ_2. On the right, note how the dissonance η_0 first needs to be mapped into its dual consonance, here ξ_5. Only forbidden steps connecting the resulting consonances are kept, drawn with thick lines. For readability, all forbidden loops have been omitted.

4.1.2 Strict Digraph Morphisms

A strict digraph associated to a counterpoint world is exclusively built out of consonances, so that the condition for preserving the allowed character of contrapuntal intervals is automatically met. We will focus on what happens to arrows. Remember that a forbidden step, and only forbidden steps, must necessarily be mapped into forbidden steps. The same holds for allowed steps.

One crucial point, specific to strict digraphs, has to be mentionned. Since a loop is attached to every vertex, the pre-image of any contrapuntal consonance ξ' in the target digraph D' under a strict digraph homomorphism ϕ must form a *clique*—also called *complete subgraph*[4]—K_m in the source digraph D. This can be a simple singleton, the trivial case K_1:

$$\begin{aligned} V(K_1) &= \{v\} \\ A(K_1) &= \{(v,v)\} \end{aligned} \tag{4.14}$$

or a complete graph $K_m, m \geq 2$ of higher order, as the one appearing at the bottom of Fig. 4.6.

$$\begin{aligned} \phi(K_m) = K_1' \subseteq D' \quad K_m \subseteq D \\ v \longmapsto v' \quad \forall v \in V(K_m) \\ (v_0, v_1) \longmapsto (v', v') \in A(D') \quad \forall v_0, v_1 \in V(K_m) \end{aligned} \tag{4.15}$$

This shows how complete subgraphs can play a key role in morphing strict graphs: they serve to lower the order of the image digraph, facilitating its embedding into another

[4] A subgraph $D = (V, A)$ equipped with all possible arrows, i.e. $A = V^2$, see [MMW04], page 99.

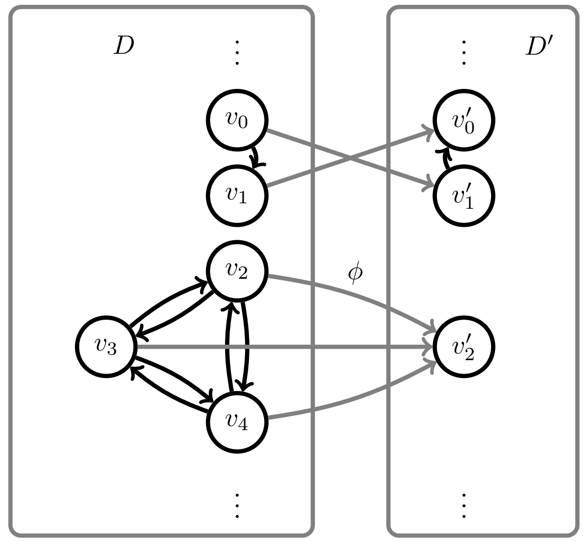


Figure 4.6: A part of a strict digraph morphism $\phi : D \to D'$ preserving arrows. The mapping is injective for the contrapuntal consonances v_0 and v_1. When this is not the case, the pre-image must be a complete subgraph. This is what happens with v_2, v_3 and v_4. All three intervals form a K_3 and are mapped onto v'_2. For readability, all forbidden loops have been omitted.

strict digraph. The absolutely remarkable property that strict digraphs associated to a counterpoint world posses a non-trivial partition into complete subgraph will be the topic of Sec. 4.2.4.

4.2 Quotient Graphs

The definition of strict digraphs reduced the dimension of the counterpoint world morphing problem by cutting the amount of vertices and arrows by a half. Quotient graphs offer a further simplification. The idea is to pack together all contrapuntal intervals showing a similar connection profile, and handle them as a single vertex. This results into much smaller and simpler graphs. The advantage lies of course on the algorithmic level: the lower the order and size of graphs, the faster the algorithms. But it is also theoretical. The *homogeneous graph* of a counterpoint world defined below is a clear, readable representation of its interdiction structure. Quotient classifications guide and organise the construction of counterpoint world morphisms.

We first define the general principle of a quotient graph, and then show how the particular constructions form an inclusion hierarchy. We will be able to walk through this classification from the coarsest down to the finest level, and the algorithm presented in Chap. 5 will follow this idea.

Notation

First of all, we introduce a simplified notation for sets of contrapuntal intervals that will help us to designate equivalence classes. A quotient component, defined below to be a vertex of a quotient graph is a set of contrapuntal intervals showing regular patterns in the cantus firmus and the interval. For any pair of reference cantus firmus $x_0 \in \mathcal{C}_n$ and cantus firmus "period" $\Delta x \in \mathcal{C}_n$, along with a set of interval classes $k_i \in \mathcal{C}_n, i \in \{1, \ldots, m\}$, we write the pattern in the following way:

$$(k_1, \ldots, k_m)_{x_0|\Delta x} := \Big\{ x + \varepsilon.y \in \mathcal{C}_n[\varepsilon] \Big| y \in \{k_1, \ldots, k_m\} \wedge x \in x_0 + \Delta x \cdot \mathcal{C}_n \Big\}. \quad (4.16)$$

Example 12. *In $\mathcal{C}_{12}$, the set of all fifth is written $(7)_{0|1}$, the set of minor and major thirds anchored on an even cantus firmus $(3,4)_{0|2}$. Remember that the numbers given are in fact residual classes modulo 12.*

Regular patterns appear only in strict digraphs associated to global counterpoint worlds, when all contrapuntal consonances are present in the vertex set. Restrictions to local counterpoint worlds may create "holes", i.e. irregularities, because of missing contrapuntal intervals. Nevertheless, we use the same notation in the local case. Local quotient components are always subsets of a unique global quotient component, so we refer to the encompassing global component by adopting this convention. Note that at the weak level, defined later in Sec. 4.2.3, we may encounter two different local components belonging to a same global component, so that both are written the same way. This is not critical, since the essential work will be done at the homogeneous level, also defined later in Sec. 4.2.5, where such collisions do not happen.

4.2.1 Quotient Graph Functor

This section will show how the definition 24 of the counterpoint world morphisms, based on total symmetry, allows a clean and unified definition of the operation of quotienting graphs.

Definition 28. *Let $D = (V, A)$ be a directed graph and $\sim$ an equivalence relation defined on its vertex set V, whose canonical projection is*

$$\begin{aligned} \pi_\sim : V &\longrightarrow V/\sim \\ v &\longmapsto [v]_\sim := \Big\{ v' \in V \Big| v' \sim v \Big\}. \end{aligned} \quad (4.17)$$

*The **quotient graph** Q of D induced by $\sim$ is defined by applying the quotient operation on its vertices and arrows.*

1. *The vertices are the equivalence classes.*

$$V(Q) := \Big\{ [v]_\sim \Big| v \in V \Big\} \quad (4.18)$$

2. *Two classes are connected as soon as there exists at least one connection between any two of its members.*

$$A(Q) := \left\{ ([v_0]_\sim, [v_1]_\sim) \middle| (v_0, v_1) \in A \right\} \tag{4.19}$$

Definition (4.19) specifies that the canonical projection is a conventional digraph homomorphism. However, it is not necessarily a *strict* digraph morphism. The image of unconnected vertices may be connected if they belong to two classes containing other members that are connected. The definition of a quotient morphism makes sense in the following situation.

Lemma 3. *Let $D = (V, A)$ and $D' = (V', A')$ be two strict digraphs, $\phi : D \to D'$ a strict digraph morphism, and $\sim$ and $\sim'$ two equivalence relations defined on V and V' respectively, such that the compatibility condition*

$$v' \sim v \Longrightarrow \phi(v') \sim_{'} \phi(v) \quad \forall v, v' \in V \tag{4.20}$$

holds. Then there exists a unique digraph homomorphism π_ϕ such that the diagram commutes.

$$\begin{tikzcd} D \arrow[r, "\phi"] \arrow[d, two heads, "\pi_\sim"'] & D' \arrow[d, two heads, "\pi_{\sim'}"] \\ D/\sim \arrow[r, dotted, "!", "\pi_\phi"'] & D'/\sim_{'} \end{tikzcd} \tag{4.21}$$

Proof. We define the mapping of equivalence classes as

$$\begin{aligned} \pi_\phi : V/\sim &\longrightarrow V'/\sim_{'} \\ [v]_\sim &\longmapsto [v']_{\sim'} = \pi_{\sim'} \circ \phi \circ \pi_\sim{}^{-1}([v]_\sim) . \end{aligned} \tag{4.22}$$

Equation (4.20) guarantees that the set function π_ϕ is well defined: a whole equivalence class $[v]_\sim$ gets indeed mapped into a single equivalence class $[\phi(v)]_{\sim'}$. This mapping is also unique. □

Note that the preservation of arrows is guaranteed by the fact that ϕ preserves arrows, and by the definition of quotient strict graphs (4.19), where an arrows exist between two classes as soon as two members are connected. This asymmetrical definition ("if at least one pair of members is connected" and not "if and only if all pairs of members are connected") does not ensure the preservation of the absence of arrows. A quotient mapping will always be a usual digraph homomorphism, but its strict character will depend on the equivalent classes involved, as discussed in Sec. 4.3.2.

In situations where the compatibility condition (4.20) holds, we can define the quotient functor $\mathcal{Q}$ that maps strict digraphs $D = (V, A)$ into their quotient digraphs Q

$$Q := \left(\pi_\sim(V), \pi_\sim^2(A)\right) \tag{4.23}$$

and strict morphisms ϕ into their quotient morphisms $\mathcal{Q}\phi$, as defined in (4.22). The *strict* character is preserved: if D is reflexive, so will be $\pi_{\sim}(D)$ and $\pi_{\sim'}(D')$.

We next introduce the five connectivity equivalence relations, from the coarsest to the finest one.

4.2.2 Null Quotient Digraphs

We will add two trivial equivalence relations to the more common ones defined in the next sections. The idea is to transform the strict digraph itself, as well as its single vertices, into subsets. A common formalism can then be used to describe them together with the intermediate equivalence classes.

Null Equivalence Relation

We begin with the broadest possible equivalence relation.

Definition 29. *The* ***null equivalence relation*** $\sim_N$ *packs all vertices of a strict digraph* D *into a single component* n.

$$v' \sim_N v \quad \forall v, v' \in V(D) \tag{4.24}$$

Reflexivity, symmetry and transitivity are trivially verified.

Null Quotient Functor

The null quotient digraph N is a complete singleton K_1. The associated functor $\mathcal{N}$: $\mathfrak{SDGph} \to \mathfrak{SDGph}$ is called **null quotient functor**.

4.2.3 Weak Quotient Digraphs

Connectivity in an undirected graph is straightforward to define. But as soon as we consider directions, the definition is not unique any more. A first one is the *weak connectivity*, inherited from the underlying undirected graph.

Definition 30. *Two vertices* v *and* v' *of a directed graph* $D = (V, A)$ *are* ***weakly connected*** *iff there exists an undirected walk* $p : \{0, \ldots, S\} \to V$ *connecting them, such that:*

$$\begin{aligned} v_0 &= v \\ v_S &= v' \\ (v_{s-1}, v_s) \in A \vee (v_s, v_{s-1}) &\in A \quad \forall s \in \{1, \ldots, S\}. \end{aligned} \tag{4.25}$$

The notation v_s *is a shorthand for* $p(s)$.

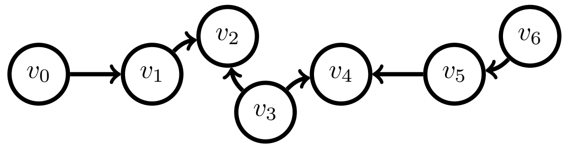


Figure 4.7: Vertex v_0 is weakly connected to vertex v_6 by means of an undirected walk.

Weak Equivalence Relation

On a strict digraph D, the weak connectivity induces the weak equivalence relation $\sim_W$:

$$v' \sim_W v :\Leftrightarrow v' \text{ is weakly connected to } v. \tag{4.26}$$

It is indeed an equivalence relation since it shows all necessary properties:

1. Reflexivity is guaranteed by lazy walks (paths of length 0). But in the special case of strict digraphs, there is also a systematic loop attached to each vertex: reflexivity is guaranteed by non trivial walks.

2. Symmetry because we ignore the orientation of the walks.

3. Transitivity because it is always possible to concatenate undirected walks sharing a common end.

Weak Quotient Functor

Definition 31. *The* ***weak quotient functor*** $\mathcal{W} : \mathfrak{SDGph} \to \mathfrak{SDGph}$ *associates the weak strict digraph* W *to a strict digraph* $D = (V, A)$*. It is defined by the weak equivalence relation (4.26).*

1. *The vertices are the weakly connected components, called* ***weak components*** *for short.*
$$V(W) := V(D) / \sim_W \tag{4.27}$$

2. *The arrow set only contains loops, making it a discrete graph.*
$$A(W) := \left\{ \left([v]_{\sim_W}, [v]_{\sim_W}\right) \middle| [v]_{\sim_W} \in V(W) \right\} \tag{4.28}$$

If a strict digraph D is not weakly connected, it means that $W \ncong K_1$. Each weak component of the partition is totally independent from the rest of the digraph, it can be treated separately. If several components are isomorphic, i.e. share the same arrow structure, the work done by investigating one component can be reused for equivalent components.

4.2.4 Strong Quotient Digraphs

The next level of connectivity is called *strong*. We do not require a connection in at least one of two directions, but in both simultaneously. This may recall the difference between disjunction and conjunction.

Definition 32. *Two vertices v and v' of a directed graph D are* ***strongly connected*** *iff there exist two directed walks $p = (v_s)_{s=0}^{S}$ and $p' = (v'_{s'})_{s'=0}^{S'}$, connecting v to v' and v' to v respectively.*

$$\begin{cases} v_0 = v \\ v'_0 = v' \\ v_S = v' \\ v'_{S'} = v \\ (v_{s-1}, v_s) \in A & \forall s \in \{1, \ldots, S\} \\ (v'_{s-1'}, v'_{s'}) \in A & \forall s' \in \{1, \ldots, S'\} \end{cases} \tag{4.29}$$

Strong Equivalence Relation

This second type of connectivity again yields an equivalence relation $\sim_S$ when defined on digraphs.

$$v' \sim_S v :\Leftrightarrow v' \text{ is strongly connected to } v \tag{4.30}$$

As before, reflexivity is guaranteed by lazy walks, symmetry by the bidirectional aspect of definition 32, and transitivity by concatenation of directed walks. Strong equivalence classes, i.e. strongly connected components, are often called *blocks* in literature.

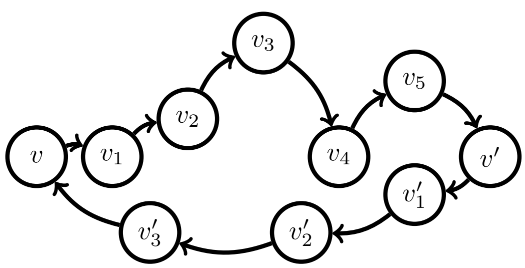


Figure 4.8: Two vertices v and v' are strongly connected by means of two different paths. v and v' then always belong to a directed cycle.

Strong Quotient Functor

Definition 33. *The* ***strong quotient functor*** *$\mathcal{S} : \mathfrak{SDGph} \to \mathfrak{SDGph}$ is the quotient functor induced by the strong equivalence relation $\sim_S$, that associates the* ***strong quotient graph*** *S of strongly connected components, or* ***strong components*** *for short, to a strict digraph $D = (V, A)$.*

Complete Blocks

The systematic presence of a loop in a strict digraph, as well as the obligation to preserve both authorisations and interdictions for a strict digraph morphism force preimages to form cliques in the source digraph.

It is a striking fact that the structure of counterpoint worlds yields associated strict digraphs whose strong blocks are also complete subgraphs. They show this additional feature that any pair of contrapuntal intervals taken from a same block is not only linked by a large circuit, as would be the case for example in a cyclic digraph, but are also directly connected in both directions.

Conjecture 2. *Let $D = (V, A)$ be a strict digraph associated to a counterpoint world $CW = (\kappa, \sigma, p^{\bullet}_{\Delta})$. All its blocks form complete subgraphs.*

It is true for $6 \leq n \leq 12$, and may be a consequence of conjecture 1, which pretends that v has to be a zero divisor. This fact induces periodicities in the cantus firmus pattern, as shown in Tab. C.1 to C.15.

4.2.5 Homogeneous Quotient Digraphs

The weak and strong equivalence relations are part of common graph theory and work for any digraph homomorphisms, in the sense that only the preservation of arrows is needed to guarantee that the quotient morphism are well defined. The fourth equivalence relation will involve the strict character and the preservation of absence of arrows. This will allow us to define groups of contrapuntal consonances showing exactly the same connection profile. Manipulations preserving the homogeneous connectivity won't alter the original strict digraph's arrow structure. A far as the preservation of arrows is concerned, homogeneous blocks can be treated as if they were single vertices, see Fig. 4.9.

Homogeneous Equivalence Relation

Definition 34. *Two vertices v and v' of a strict digraph D are **homogeneously connected** if they share the same set of predecessors $N^{(0)}_D(\cdot)$ and successors $N^{(1)}_D(\cdot)$, i.e. if they are connected to the same set of vertices, inducing an equivalence relation on strict digraphs. Two vertices v and v' belong to the same homogeneity class iff*

$$v' \sim_H v :\Leftrightarrow \begin{cases} N^{(0)}_D(v) = N^{(0)}_{D'}(v') \\ N^{(1)}_D(v) = N^{(1)}_{D'}(v') \,. \end{cases} \tag{4.31}$$

This relation is indeed an equivalence relation:

1. Reflexivity. A vertex trivially shares its own connection profile with itself.

2. Symmetry. Equations (4.31) are perfectly symmetric.

3. Transitivity. The equalities in (4.31) are also transitive.

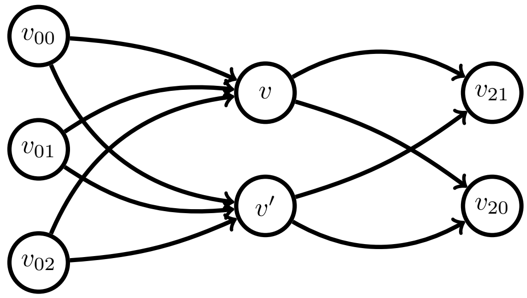


Figure 4.9: Homogeneous connectvity: vertices v and v' share exactly the same predecessors $\{v_{00}, v_{01}, v_{02}\}$ and successors $\{v_{10}, v_{11}\}$.

Homogeneous Quotient Functor

It is easily verified that a strict digraph morphism preserves the connectivity profile,[5] since absence and presence of arrows is preserved. The homogeneous equivalence is hence compatible in the sense of (4.20) and allows the definition of its associated quotient functor.

Definition 35. *The* ***homogeneous quotient functor*** $\mathcal{H} : \mathfrak{SDGph} \rightarrow \mathfrak{SDGph}$ *is the quotient functor induced by the homogeneous equivalence relation* $\sim_H$ *on the category of strict digraphs* $\mathfrak{SDGph}$.

4.2.6 Full Quotient Digraphs

We will finish this enumeration with the finest possible equivalence relation, the trivial relation that splits all elements into single classes.

Full Equivalence Relation

Definition 36. *The* ***full equivalence relation*** $\sim_F$ *packs each vertex of a strict digraph* D *into a single component* f.

$$v' \sim_F v :\Leftrightarrow v' = v \quad \forall v, v' \in V(D) \tag{4.32}$$

It is trivially verified that it is indeed an equivalence relation.

Full Equivalence Functor

The full quotient digraph F is the exact replica of the original strict digraph D, except that every vertex has been replaced by a singleton subset containing it.

$$v \mapsto \{v\} \tag{4.33}$$

The associated functor $\mathcal{F} : \mathfrak{SDGph} \rightarrow \mathfrak{SDGph}$ is called **full quotient functor**.

[5] As long as we consider only the image digraph, and not the whole codomain, which may possibly contain more vertices and arrows, see the discussion about strong quotient mappings in Sec. 4.3.2.

4.3 Quotient Hierarchy

It is clear that the strong equivalence yields a partition that is a refinement of the weak partition. If two vertices are strongly connected, they are necessarily weakly connected.

$$v' \sim_S v \Rightarrow v' \sim_W v \quad \forall v, v' \in V \tag{4.34}$$

In the general case, strong and homogeneous partitions are independent. But the latter is a refinement of the former in the special case of a strict digraph $D = (V, A)$. This a consequence of the systematic existence of loops, which always forces a vertex to belong to its own neighbourhood:

$$v \in N_D^{(1)}(v) \quad \forall v \in V \tag{4.35}$$

Since two homogeneously equivalent vertices $v \sim_H v'$ share the same neighbourhood, $N_D^{(1)}(v) = N_D^{(1)}(v')$, we have

$$v \in N_D^{(1)}(v'). \tag{4.36}$$

Permute v and v', and the converse is true: $v \in N_D^{(1)}(v')$, so that both vertices are directly connected and belong to the same strong component.

$$v' \sim_H v \Rightarrow v' \sim_S v \quad \forall v, v' \in V \tag{4.37}$$

4.3.1 Quotient Component inclusions

The five equivalence relations represent successive refinements of their partitions. We first give a formal definition for an inclusion relation between two quotient components.

Definition 37. *Let $D = (V, A)$ be a strict digraph and $\mathcal{Q}$ and $\mathcal{P}$ be a pair of quotient functors chosen among the five functors $\mathcal{N}, \mathcal{W}, \mathcal{S}, \mathcal{H}, \mathcal{F}$ defined previously. Let $Q = \mathcal{Q}(D)$ and $P = \mathcal{P}(D)$ be their respective quotient digraphs. We say that the quotient functor $\mathcal{Q}$ is* ***included*** *in $\mathcal{P}$, or is a* ***refinement*** *of $\mathcal{P}$ in the following situation:*

$$\mathcal{Q} \subseteq \mathcal{P} :\Leftrightarrow \forall q \in V(Q), \exists! p \in V(P) : {\pi_{\sim_P}}^{-1}(p) \supseteq {\pi_{\sim_Q}}^{-1}(q) \tag{4.38}$$

This relationship is illustrated by the following diagram:

$$\begin{tikzcd} p \arrow[r, mapsto, "{\pi_{\sim_P}}^{-1}"] & D[p] \\ q \arrow[r, mapsto, "{\pi_{\sim_Q}}^{-1}"'] & D[q] \arrow[u, hook] \end{tikzcd} \tag{4.39}$$

where the notational shorthand for the induced subdigraph $D[q]$ stands for the cleaner but cumbersome expression $D[{\pi_{\sim_Q}}^{-1}(q)]$. Tab. 4.3 orders all equivalence relations

Table 4.3: The different levels i, their quotient functor $\mathcal{Q}^{(i)}$, name and components q. The numbering starts with the broadest level (i=0) and increases up to the finest level (i=4).

i	$\mathcal{Q}^{(i)}$	Name	q
0	$\mathcal{N}$	Null	Whole strict digraph
1	$\mathcal{W}$	Weak	Weak components
2	$\mathcal{S}$	Strong	Strong blocks
3	$\mathcal{H}$	Homogeneous	Homogeneous components
4	$\mathcal{F}$	Full	Contrapuntal intervals

from the broadest partition down to the finest one, in the sense that each component (equivalence class) of a given level i is entirely contained inside a single component of a previous (broader) level.

In Chap. 5, level 0, the null quotient, will serve as a root for all tree structures, and hence as a starting point for all recursive procedures propagating down the quotient hierarchy until they eventually reach the contrapuntal interval level.

4.3.2 Mapping Quotient Components

The special nature of a strict digraph morphisms ϕ imposes certain restrictions on the freedom of mapping quotient components between two strict digraphs $D = (V, A)$ and $D' = (V', A')$.

1. As a general property of digraph homomorphisms, two members $v_0, v_1 \in V$ of a same quotient component q cannot be mapped into two different quotient components $q'_0, q'_1 \in V(Q') : q'_0 \neq q'_1$ without breaking, i.e. not preserving, a connection.

$$(v_0, v_1) \in A \Rightarrow (\phi(v_0), \phi(v_1)) \in A' \quad \forall v_0, v_1 \in V \tag{4.40}$$

   Thus, the image q' of a quotient component q under $\mathcal{Q}\phi$ is well defined.

2. As a particular property of strict digraph morphisms, quotient morphisms have to be injective in most cases, since they have to preserve the absence of arrows.

$$(v_0, v_1) \notin A \Rightarrow (\phi(v_0), \phi(v_1)) \notin A' \tag{4.41}$$

   Two exceptions to these rules are weak and full components. In which circumstances exceptions to injectivity may occur is described below, as well as another, more subtle, exception that appears at the strong level, where it is sometimes necessary to consider only the image digraph and not its embedding global target digraph.

These *horizontal* restrictions imply also *vertical* restrictions, in the sense that the choice of mappings at broader and finer levels are not independent. Let again $\mathcal{Q}$ and $\mathcal{P}$ be two quotient functors such that $\mathcal{Q} \subseteq \mathcal{P}$, $D = (V, A)$ and $D' = (V', A')$ be two strict digraphs, Q, Q', P and P' their respective quotient digraphs.

1. Upwards: mapping quotient component $q \in V(Q)$ onto $q' \in V(Q')$ at a given level $\mathcal{Q}$ will determine a unique mapping at any broader level $\mathcal{P} \supseteq \mathcal{Q}$.

$$q' = \mathcal{Q}\phi(q) \Rightarrow p' = \mathcal{P}\phi(p) \tag{4.42}$$

   where

$$\begin{cases} \exists! p \in V(P) : p \supseteq q \\ \exists! p' \in V(P') : p' \supseteq q' \end{cases} \tag{4.43}$$

   Here again, we simplify the notation by writing $p \supseteq q$ instead of ${\pi_{\sim_P}}^{-1}(p) \supseteq {\pi_{\sim_Q}}^{-1}(q)$.

2. Downwards: a broader connection may restrict the choice of finer ones, but this time, the choice is not necessarily unique. If we have

$$p' = \mathcal{P}\phi(p) \tag{4.44}$$

   then any subquotient component $q \in V(Q) : q \subseteq p$ of the source component p may be mapped to a subquotient component of the target component p':

$$q' = \mathcal{Q}\phi(q) \subseteq p' \tag{4.45}$$

   Whether such an assignment yields a valid quotient morphism or not is discussed in Chap. 5, where the algorithm for enumerating all strict morphism digraphs will rely heavily on this phenomenon of vertical guiding, or channelling.

We now show how the horizontal containment rules (4.40) and (4.41) apply at each level.

Null Quotient Mappings

Null quotient components are unique, so the mapping is trivial: a singleton onto a singleton.

Weak Quotient Mappings

Weak quotient digraphs are discrete strict digraphs. An arbitrary mapping will always preserve connections so that a weak quotient morphism $\mathcal{W}\phi$ is a usual graph homomorphism.

The morphism is however not necessarily strict: two different (and disconnected) weak components w_0 and w_1 in D can be mapped into the same (and self-connected) target weak block as long as their image do not *touch*, i.e. do not share the same neighbourhood, as can be seen from Fig. 4.10.

$$\begin{cases} N_{D'}(\phi(w_0)) \cap \phi(w_1) = \emptyset \\ N_{D'}(\phi(w_1)) \cap \phi(w_0) = \emptyset \end{cases} \tag{4.46}$$

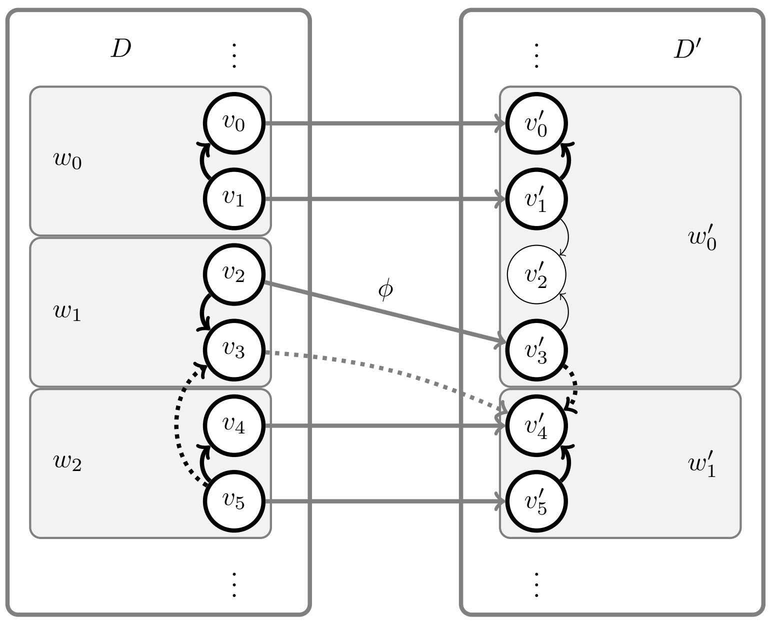


Figure 4.10: An example of how the weak containment rule can be broken. If we map v_3 onto v'_4, forbidden steps (v_5, v_3) as well as (v'_3, v'_4) would be necessary for preservation, but do not exist. On the other hand, v'_2 plays a buffer role that allows v_2 to be mapped into v'_3 even if w_0 is already mapped into w'_0.

Strong Quotient Connections

Here too, the strong blocks act as a guide: the strict digraph morphisms can't split a source strong block into several target strong blocks without breaking or creating connections, so that the strong quotient morphism $\mathcal{S}\phi$ is well defined and injective. See Fig. 4.11 for an illustration.

Strong blocks being maximally complete subgraphs, the images of two different ones would necessarily be connected if they would be mapped into the same strong component.

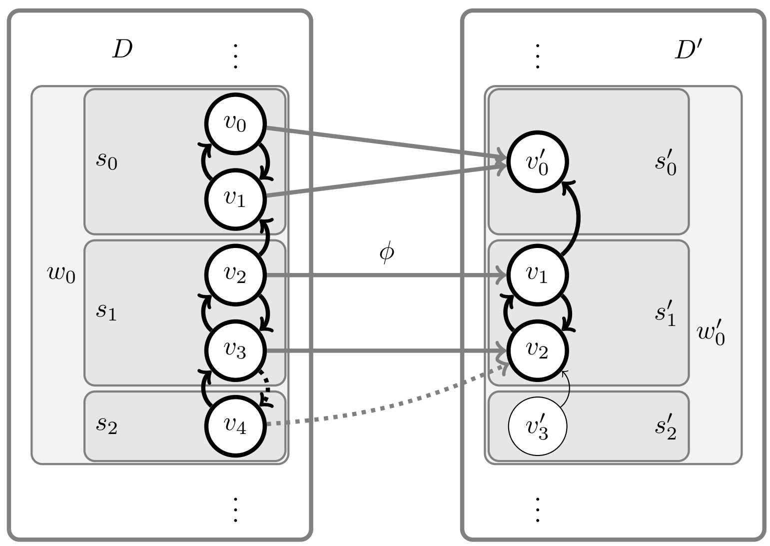


Figure 4.11: An example of how the strong containment rule would be broken. A pre-image is a clique and must be contained inside a single strong component, or forbidden steps will be missing, like (v_3, v_4).

If we consider strong blocks on the full target strict digraph instead of its restriction to the image set $S[\mathrm{Im}\,(\mathcal{S}\phi)]$, the strong quotient morphism $\mathcal{S}\phi$ may not be strict any more. Fig. 4.12 illustrates this problem, which is peculiar to the strong level and does not happen with weak or homogeneous quotient digraphs.

Homogeneous Quotient Connections

The homogeneous quotient component morphism is strict and injective, due to the necessity of preserving arrows. Only vertices with an identical connection profile can be freely mapped and shuffled without breaking or adding any connection, see Fig. 4.13.

The essential structure of a counterpoint world can be represented with a homogeneous quotient graph drawing. The ability to visualise counterpoint worlds is one of the very important results of the application of graphs to counterpoint theory: Fig. 4.15 shows this for the six worlds in $n = 12$. Note how the different choices of consonances

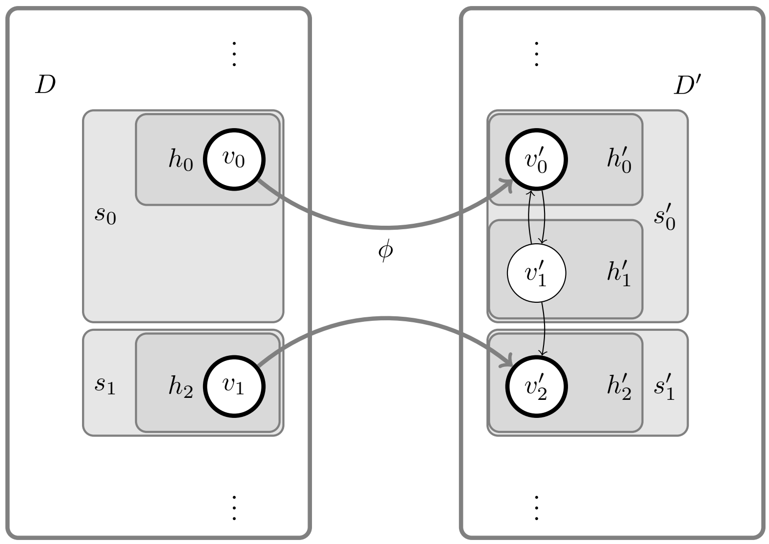


Figure 4.12: The strong quotient morphism is strict only if we consider its image, not the entire global codomain. As long as h_0 does not get mapped onto h'_1, the connection structure is preserved at the strong level. While s'_0 is connected to s'_1 in the global target graph, this is not the case in the graph restricted to the image set: the connection originating from h'_1 then disappears, and so does the connection at the strong level. In order to verify the preservation of strong connectivity from the domain to its image under $\mathcal{S}\phi$, it is necessary to know the image set completely, or at least the neighbourhood of strong components containing more than one homogeneous component. As will be described in Chap. 5, this information may not be available before the construction of a morphism is complete.

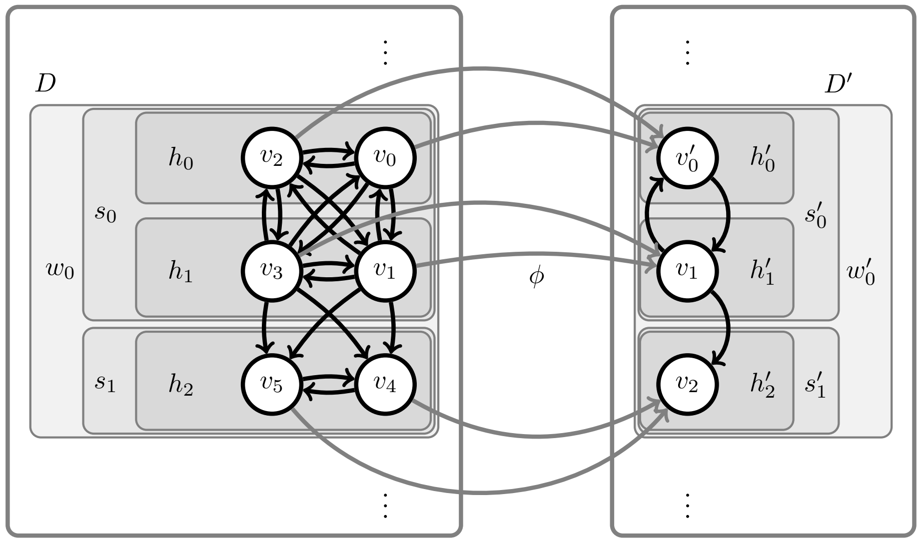


Figure 4.13: Homogeneous mapping is both strict and injective. It faithfully preserves the arrow structure between strict digraphs.

shape completely different structures, even if all six graphs have been generated with the same algebraic procedure.

Full Quotient Connections

The homogeneous quotient components have been defined in such a way that every vertex of a homogeneous block can be mapped arbitrarily to any vertex of a target homogeneous block, as long as the mapping is valid at the homogeneous level, see Fig. 4.14. These mappings are strict, but not necessarily injective.

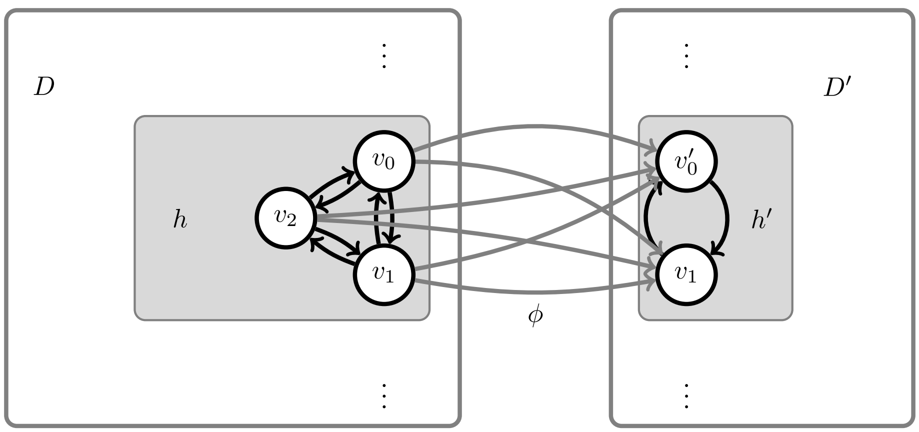


Figure 4.14: A full mapping does not have to be injective. Every assignment between vertices belonging to blocks connected by a valid homogeneous mapping is valid.

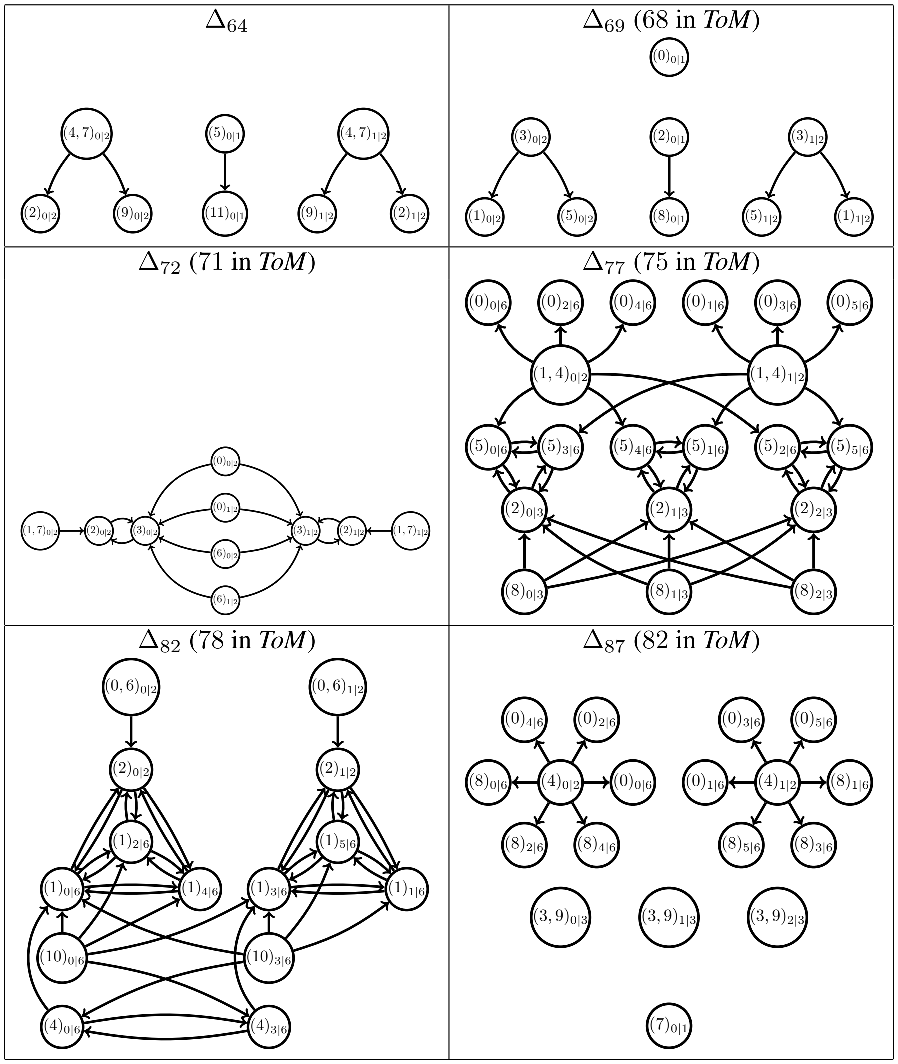


Figure 4.15: The homogeneous quotient graphs of the six representative counterpoint worlds in $\mathcal{C}_{12}$. The number indicate the affine class index of the strong dichotomy, and correspond to those appearing in Tab. 2.2. For readability, loops attached to each vertex are omitted. In world Δ_{82}, the structures attached to the bottom of the $(1)_{\cdot|6}$ vertices appear only once. The other two have been omitted and show a $(\cdot)_{1|6}$/$(\cdot)_{4|6}$ respectively $(\cdot)_{2|6}$/$(\cdot)_{5|6}$ periodicity. The designation of vertices follows the notation for quotient components introduced in (4.16).

4.4 Scales

Rules of counterpoint apply to relative movements and intervals between voices. They tell nothing about the usage of a particular, absolute, pitch class. A scale dictates which pitch classes are allowed: those belonging to its degrees. The rest should be avoided.

Definition 38. *A **scale** $\mathcal{S}$ is an unordered subset of the chromatic circle*

$$\mathcal{S} \subseteq \mathcal{C}_n. \tag{4.47}$$

The scale indicator function

$$\begin{aligned} \mathbb{1}_{\mathcal{S}} : \mathcal{C}_n : &\longrightarrow \{\top, \bot\} \\ k &\longmapsto \begin{cases} \top & k \in \mathcal{S} \\ \bot & \text{otherwise} \end{cases} \end{aligned} \tag{4.48}$$

says if a particular pitch class k belongs to the scale's degrees or not.

Example 13. *The diatonic scale $\mathcal{S}$ in $\mathcal{C}_{12}$. Semitones are characterised by the succession of two true values ($\top$) of the indicator function.*

Pitch class	k	$\mathbb{1}_{\mathcal{S}}(k)$
C	$[0]$	$\top$
$C\sharp$	$[1]$	$\bot$
D	$[2]$	$\top$
$D\sharp$	$[3]$	$\bot$
E	$[4]$	$\top$
F	$[5]$	$\top$
$F\sharp$	$[6]$	$\bot$
G	$[7]$	$\top$
$G\sharp$	$[8]$	$\bot$
A	$[9]$	$\top$
$A\sharp$	$[10]$	$\bot$
B	$[11]$	$\top$

It is possible to superpose the restrictions induced by scales to the rules of counterpoint. This introduces a new dichotomy, a division of pitch classes into allowed ones and forbidden ones, that has a direct impact on the possibility to use contrapuntal consonances.

Definition 39. *An oriented contrapuntal consonance $\hat{\xi} = (x + \varepsilon.y, \Omega)$ is said to be **compatible** with a scale $\mathcal{S}$ for a given orientation Ω if both its cantus firmus x and projected discantus $z = \Omega(x + \varepsilon.y)$ belong to the scale.*

$$\mathbb{1}_{\mathcal{S}}(x) \wedge \mathbb{1}_{\mathcal{S}}(z) = \top \tag{4.49}$$

The set of all oriented contrapuntal consonances $\hat{\xi}$ in $\hat{K}[\varepsilon]$ compatible with a scale $\mathcal{S}$ will be denoted by $\hat{K}_{\mathcal{S}}[\varepsilon]$.

We already restricted a global counterpoint world to the consonances used by a counterpoint to form a local counterpoint world. Here we apply the same idea quite differently to select only those oriented contrapuntal consonances compatible with a given scale. This is another kind of local restriction. It will model the possibilities left open for musical composition.

Note that there is one subtlety which needs to be handled carfully: the orientation. In a counterpoint world, the consonant character of a contrapuntal interval is not affected by its sweeping or hanging orientation. However, as soon as we introduce scales, the position of the discantus becomes crucial. For a given cantus firmus and interval, the compatibility of the discantus with the scale may depend on the orientation. For example, in case of a sweeping major third $([0] + \varepsilon.[4], \Omega_+)$, the discantus is located on pitch class $[4]$, an E, which belongs to the diatonic scale, as listed in example 13. But if we flip the same interval to its hanging orientation Ω_-, the discantus moves to $[8]$, an $A\flat$ not included in the scale. We thus can't ignore the orientation any more, as we did in the previous sections, when we were considering counterpoint rules only.

In order to describe how a scale restricts the counterpoint rules, an extended kind of digraph has to be defined, which takes orientation into account.

Definition 40. *The oriented authorisation digraph $\hat{D}$ associated to a counterpoint world $CW = (\kappa, \sigma, p^\bullet_\Delta)$ is a directed graph whose vertices are the oriented contrapuntal consonances, and the arrows the allowed steps.*

$$\begin{aligned} V\left(\hat{D}\right) &:= \hat{K}[\varepsilon] = \left\{(\zeta, \Omega) \in \hat{\mathcal{C}}_n[\varepsilon] \middle| \kappa(\zeta) = \top\right\} \\ A\left(\hat{D}\right) &:= \left\{((\xi_0, \Omega_0), (\xi_1, \Omega_1)) \in \hat{K}[\varepsilon] \times \hat{K}[\varepsilon] \middle| \sigma(\xi_0, \xi_1) = \top\right\} \end{aligned} \tag{4.50}$$

Taking the orientation into account doubles the number of vertices: each contrapuntal consonance now appears twice, in its hanging and sweeping orientations.

The subdigraph $\hat{D}[\hat{K}_{\mathcal{S}}[\varepsilon]]$ induced by the set of compatible contrapuntal consonances $\hat{K}_{\mathcal{S}}[\varepsilon]$ models the steps available for composition compatible with both scale and counterpoint rules. Comparing the parameters of these two digraphs gives a broad average measure of their compatibility degree:

1. The ratio of the graph orders (number of vertices) tells which proportion of contrapuntal consonances are still available after the pitch classes have been restricted to the pool of those authorised by the scale.

$$\nu(CW, \mathcal{S}) := \frac{n\left(\hat{D}[\hat{K}_{\mathcal{S}}[\varepsilon]]\right)}{n\left(\hat{D}\right)} \tag{4.51}$$

2. The ratio of the graph sizes (number of arrows) tells us which proportion of allowed steps are still available between compatible contrapuntal consonances.

$$\alpha(CW, \mathcal{S}) := \frac{a\left(\hat{D}[\hat{K}_{\mathcal{S}}[\varepsilon]]\right)}{a\left(\hat{D}\right)} \tag{4.52}$$

Example 14. *Pentatonic and heptatonic scales are commonly used in the $\mathcal{C}_{12}$ context. Fig. D.1 to D.8 show the relative losses defined in (4.51), respectively (4.52), for each scale class. Tab. 4.4 summarises which scale is the most or less compatible with each counterpoint world. For the pentatonic scales, the available pool of consonances drops down to 10–30% and the legal steps to less than 7%, while more than 40% of consonances can be used with heptatonic scales and less than 20% of steps.*

Table 4.4: Best and worst compatibility degrees for pentatonic and heptatonic scales with each of the $\mathcal{C}_{12}$ counterpoint worlds Δ. Ratio of compatible consonances ν and compatible steps α, along with the dihedral class index $[\mathcal{S}]_{\mathbb{D}_{12}}$ of the corresponding scale $\mathcal{S}$. Index 1 corresponds to the pentatonic, respectively diatonic scale, and index 4 to the the unitonic scale and its five note complement. Higher indices represent more exotic scales: 11 contains the K^*-scale found on page 658 of *ToM* [MGM02]. Strong dichotomy class numbers correspond to 64, 68, 71, 75, 82 in *ToM*.

Pentatonic scales								
	Consonances				Contrapuntal steps			
	Minimum		Maximum		Minimum		Maximum	
Δ	ν	$[\mathcal{S}]_{\mathbb{D}_{12}}$	ν	$[\mathcal{S}]_{\mathbb{D}_{12}}$	α	$[\mathcal{S}]_{\mathbb{D}_{12}}$	α	$[\mathcal{S}]_{\mathbb{D}_{12}}$
Δ_{64}	0.111	4	0.194	1	0.004	4	0.036	1
Δ_{69}	0.181	4	0.208	1	0.023	4	0.042	24
Δ_{72}	0.153	11	0.222	7	0.022	11	0.053	22
Δ_{77}	0.139	7	0.236	4	0.019	7	0.056	11
Δ_{82}	0.153	11	0.292	4	0.023	11	0.070	4
Δ_{87}	0.139	22	0.264	11	0.017	22	0.068	11

Heptatonic scales								
	Consonances				Contrapuntal steps			
	Minimum		Maximum		Minimum		Maximum	
Δ	ν	$[\mathcal{S}]_{\mathbb{D}_{12}}$	ν	$[\mathcal{S}]_{\mathbb{D}_{12}}$	α	$[\mathcal{S}]_{\mathbb{D}_{12}}$	α	$[\mathcal{S}]_{\mathbb{D}_{12}}$
Δ_{64}	0.278	4	0.361	1	0.062	4	0.129	1
Δ_{69}	0.347	4	0.375	1	0.108	4	0.139	24
Δ_{72}	0.319	11	0.389	7	0.100	11	0.157	22
Δ_{77}	0.306	7	0.403	4	0.093	7	0.164	11
Δ_{82}	0.319	11	0.458	4	0.100	11	0.186	4
Δ_{87}	0.306	22	0.431	11	0.089	22	0.185	11

The abundance of interdictions caused by the combination of both counterpoint and scale rules may vary, in some cases leading to total incompatibilities, i.e. situations for which it is not possible to move further from a given contrapuntal interval without either leaving the scale, using a dissonance, or making a forbidden step. Such situations cause compositional dead-ends[6], see Tab. 4.5 for an enumeration of such cases.

[6]The *ToM* [MGM02] uses the French term *cul-de-sac* in Sec. 31.

Table 4.5: Compositional dead-ends resulting from the usage of pentatonic scales in conjunction with counterpoint rules from the DUR (Δ_{64}) and FUX (Δ_{87}) worlds respectively. Counterpoint world $[\Delta]_{\mathbb{I}_{12}}$, scale index $[\mathcal{S}]_{\mathbb{D}_{12}}$ and the eight oriented contrapuntal consonances without successors. Plus and minus signs serve to designate the orientation. Scale index 4 stands for the pentatonic complement of the unitonic scale, which is a whole tone scale augmented by one note in-between. Index 5 stands for the pentatonic complement of the double harmonic scale, or mela nr. 15 in raga music. Other pentatonic or heptatonic scales combined with counterpoint worlds do not lead to such situations in $\mathcal{C}_{12}$.

$[\Delta]_{\mathbb{I}_{12}}$	$[\mathcal{S}]_{\mathbb{D}_{12}}$	$\left\{\hat{\zeta} \in V\left(\hat{D}[\hat{K}_{\mathcal{S}}[\varepsilon]]\right) \middle\vert a^{(1)}\left(\hat{\zeta}\right) = 0\right\}$
Δ_{64}	4	$[0] \pm \varepsilon.[4], [2] - \varepsilon.[4], [4] \pm \varepsilon.[4], [8] \pm \varepsilon.[4], [10] + \varepsilon.[4]$
Δ_{87} (82 in *ToM*)	5	$[0] \pm \varepsilon.[4], [2] - \varepsilon.[4], [4] \pm \varepsilon.[4], [8] \pm \varepsilon.[4], [10] + \varepsilon.[4]$

4.5 Summary

A counterpoint world naturally translates into a directed graph: vertices are contrapuntal consonances and arrows forbidden steps. The focus is on the interdictions, as illustrated by Fig 4.2. Such a strict digraph contains the entire contrapuntal structure of a world. Transforming counterpoints can then be achieved by transforming their associated digraphs. Preserving the connectivity structure guarantees that the contrapuntal structure will not be altered.

Figure 4.16: Traffic signs as an analogy to strict digraphs. A counterpoint world (the rules) is like a city, and a counterpoint (the musical piece) like a drive along the roads of this city. Crossings form the vertices (consonances). They are sometimes connected by one-way roads, which form the arrows (forbidden steps).

Strict digraph morphisms are a more severe version of the usual graph homomorphisms: the absence of arrows should also be preserved. The image of a strict digraph morphism is thus isomorphic to its pre-image, as shown in Fig. 4.17. This problem is known in the literature as the *Isomorphic Subgraph Problem* (ISP).

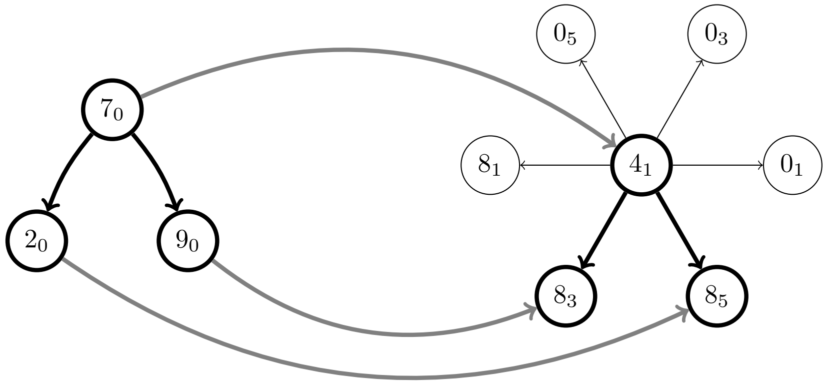


Figure 4.17: Since a strict digraph faithfully preserves the connection structure of graphs, the image has to be isomorphic to the pre-image. This example shows one possibility out of $6{\cdot}5 = 30$ different ways to embed a 2-branched star taken from world Δ_{64} into a Fuxian 6-branched star. Numbers are residual classes modulo 12.

Strict digraphs associated to counterpoint worlds show the nice property that *blocks*, or strongly connected components, always form complete subgraphs, as seen from Fig. 4.18. This may be a consequence of Conj. 1 of Sec. 2.2.2 and is the topic of Conj. 2.

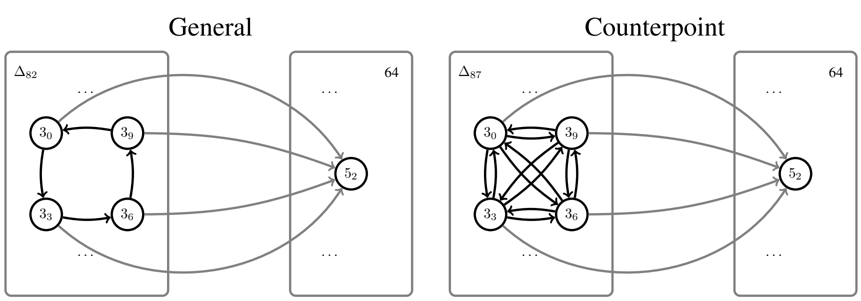


Figure 4.18: While strong blocks of general digraphs are made out of cycles, in the case of strict digraphs associated to counterpoint worlds, they always form complete subgraphs. This allows entire blocks of e.g. minor thirds of the Fuxian world Δ_{87} (82 in *ToM*) to be mapped onto a single interval, here a fifth in the Dur world Δ_{64}. For readability, loops are not represented, and numbers are residual classes modulo 12.

The special character of strong blocks allows the definition of five different partitions of the vertex set. They form successive refinements and yield more simplified quotient graphs. Their hierarchical organisation is depicted inFig. 4.19.

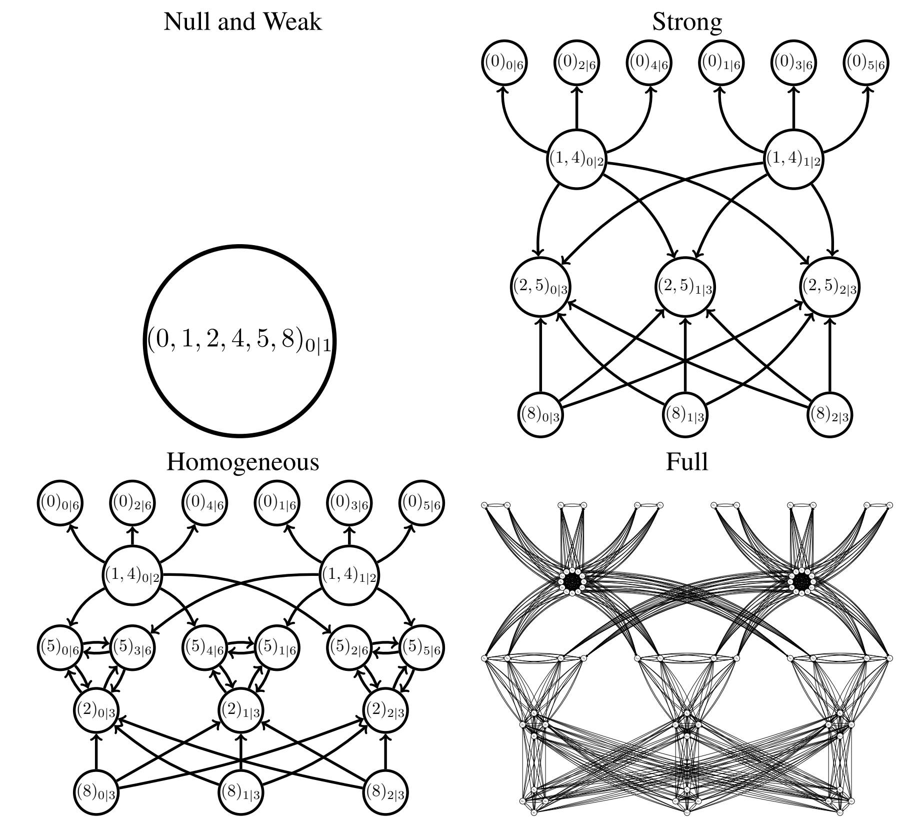


Figure 4.19: Quotient hierarchy of world Δ_{77} (75 in *ToM*). Because the strict digraph is connected, the null quotient does not split into different weak components. But the strong components $(2,5)_{\cdot|3}$ each split into a complete subgraph of three homogeneous components due to different connectivity profiles. Note how the quotient graphs are much simpler versions of the entire full graph (isomorphic to the original strict digraph) with its 72 vertices and its nearly thousand arrows. For readability, loops are not displayed. The designation of vertices follows the notation for quotient components introduced in (4.16).

Chapter 5

Morphism Enumeration

We now turn to a procedure for computing all possible counterpoint world morphisms ψ from a given counterpoint world CW into another (possibly the same) counterpoint world CW'. Both can be local or global.

This is typically a combinatorial search problem. It can be solved systematically by enumerating possible assignments of contrapuntal intervals, and then selecting only those defining valid counterpoint world morphisms. A naive approach would imply going through all of the $(n^2)^{n^2}$ mappings, and to verify each time if the images of n^2 pairs of contrapuntal intervals under each mapping are properly connected. This is absolutely not feasible, even for moderate values of n, as is the case for the usual value $n = 12$.

One solution consists in using a backtracking strategy, i.e. trying to prune the search tree early enough to maintain the enumeration and verification task to a reasonable extent. We can know in advance that some assignments will not preserve the connectivity structure, so they can safely be removed from the list of mappings needing verification. In Chap. 4, we introduced the quotient graph hierarchy and showed how mappings of finer partitions are constrained by mappings of broader partitions.

Founding the algorithm upon these *horizontal* and *vertical* constraints has two benefits. Following the inclusions filters out invalid assignments and avoids spending computational time in dead ends of the search tree. The second point is an organisation of the morphisms into a tree structure reflecting these inclusion relations.

The idea is to walk down the quotient hierarchy, computing all weak quotient morphisms, then all homogeneous quotient morphisms they induce, and so on until we eventually reach the level of actual contrapuntal interval mappings.

By doing so, we will have to find a compromise between working at the highest possible level, where discarding an invalid assignment has the greatest impact on the pruning of the search tree, and the need for digging deeper down into the hierarchy to verify the validity of assignments made at higher levels. Note that once we have reached the homogeneous quotient level, the work is almost done, since every mapping of contrapuntal intervals compatible with the homogeneous quotient morphisms will be valid. The structure of counterpoint worlds is completely described at that level.

Further addition of full assignments can be computed on the fly. Enumerating all mappings of contrapuntal intervals would just contribute to a tremendous growth of the task of enumerating quotient morphisms.

The complete procedure for morphing real scores of counterpoints is divided into three steps:

1. Transform real pitches into contrapuntal intervals.
2. Morph contrapuntal intervals.
3. Transform contrapuntal intervals back into real pitches.

Data structures and notations used by the algorithm are introduced in Sec. 5.1. The core of the algorithm, i.e. morphing counterpoints, corresponds to the above mentioned phase 2: Sec. 5.2 explains step by step how to enumerate quotient graph morphisms. This constitutes the hard work and is peculiar to the counterpoint morphism problem. Once a strict digraph morphism has been found, it can be quickly used to construct a counterpoint world morphism, as shown in Sec. 5.3. This is also where phases 1 and 3 are described. They constitute a more general problem, that of mapping members into classes and vice-versa.

The software described in Chap. 7 implements this procedure. Each step is realised by an appropriate module, one responsible for morphing counterpoints, the *BollyMorpher* presented in Sec. 7.2.6, and the two others for transforming musical scores into counterpoints (*Counterpointiser*, see Sec. 7.2.2) and counterpoints back into scores (*DeCounterpointiser*, see Sec. 7.2.3).

5.1 Data Structures

Quotient components, assignments and morphisms can be organised into tree structures reflecting the quotient hierarchy, as summarised in Tab. 4.3.

These trees will serve for all recursive subroutines of the morphism enumeration procedure. Arrows in the trees always connect elements from successive neighbour levels: null to weak, weak to homogeneous and finally homogeneous to full.

We deliberately skip the strong level in this procedure. While it plays an essential role in the development of the theory, it is cumbersome to use in practice. Most of the time, it is redundant with the homogeneous components. And when the two structures differ, things can get worse: See Fig. 4.12 for a simple example of a configuration causing problems. Such difficulties can be avoided only if we know the complete mapping in advance, which will not be the case, since the whole procedure consists in progressively pasting pieces of "local" mappings together.

Quotient Components Trees

A first example of a tree structure is given by the basic objects handled by the algorithm, namely the four types of quotient components (vertices of quotient graphs) introduced in Sec. 4.2. In order to simplify the notation, quotient components and digraphs will always be written with the same letter as their functor: In case of the generic quotient functor, $Q := \mathcal{Q}D$, $q \in V(Q)$, and the same holds for the specific functors $\mathcal{N}, \mathcal{W}, \mathcal{H}, \mathcal{F}$.

Definition 41. *The **quotient component tree** $T_{\mathcal{Q}D}$ of a strict digraph D is a rooted, directed tree.*

1. *The vertex set contains all quotient components.*

$$V(T_{\mathcal{Q}D}) := \left\{ q \in V(Q) \middle| \mathcal{Q} \in \{\mathcal{N}, \mathcal{W}, \mathcal{H}, \mathcal{F}\} \right\} \tag{5.1}$$

2. *The arrow set connects two quotient components p and q if they are related by an inclusion relation (see Def. 37) in the original strict digraph*

$$(p, q) \in A(T_{\mathcal{Q}D}) \Leftrightarrow q \subseteq p \tag{5.2}$$

and if their functors are direct successors in the refinement hierarchy

$$\begin{cases} \mathcal{P} := \mathcal{Q}^{(i)} \\ \mathcal{Q} := \mathcal{Q}^{(i+1)} \end{cases} \tag{5.3}$$

for $i \in \{0, 1, 2\}$ respectively, after having renumbered the levels to take into account the omission of the strong level: 2 *now means* homogeneous*, not* strong*, and* 3 full*, not* homogeneous*. Note that the arrows in the hierarchy tree go from bigger to smaller, the opposite direction of inclusion arrows.*

3. *The root is the unique vertex n of the null quotient graph N, containing the strict digraph itself. It is located at level 0.*

Quotient Assignment Tree

The next objects we will consider are individual assignments between source and target quotient components of a same level $\mathcal{Q} = \mathcal{Q}^{(i)}$ where $i \in \{0, 1, 2, 3\}$.

$$(q, q') \in V(Q) \times V(Q') \tag{5.4}$$

They form the building blocks of a complete quotient morphism $\mathcal{Q}\phi : Q \to Q'$:

$$\mathcal{Q}\phi := \uplus_{q \in V(Q)} (q, q') \tag{5.5}$$

where each $q' = \mathcal{Q}\phi(q)$. These assignemnts can be organised into a tree structure similar to the quotient components tree $T_{\mathcal{Q}D}$.

Definition 42. *The **quotient assignment tree** $T_{\mathcal{Q}a}$ between a pair of strict digraphs D and D' is a rooted, directed tree.*

1. *The Vertex set $V(T_{\mathcal{Q}a})$ will contain all individual assignments appearing in at least one quotient morphism*

$$V(T_{\mathcal{Q}a}) = \left\{ (q, q') \in \cup_{\mathcal{Q}} (V(Q) \times V(Q')) \middle| \mathbb{1}_{\mathcal{Q}a}((q, q')) = \top \right\} \tag{5.6}$$

where the indicator function

$$\mathbb{1}_{\mathcal{Q}a} : \cup_{\mathcal{Q}} \left(V\left(Q\right) \times V\left(Q'\right) \right) \longrightarrow \{\top, \bot\} \tag{5.7}$$

tells us if an assignment belongs to a quotient morphism or not.

$$\mathbb{1}_{\mathcal{Q}a}(q, q') = \top \Leftrightarrow \exists \mathcal{Q}\phi : Q \to Q' : q' = \mathcal{Q}\phi\left(q\right) \tag{5.8}$$

2. *The arrow set connects each pair of successive quotient assignments compatible with inclusion:*

$$((p, p'), (q, q')) \in A\left(T_{\mathcal{Q}a}\right) \Leftrightarrow \begin{cases} q \subseteq p \\ q' \subseteq p' \end{cases} \tag{5.9}$$

where $\mathcal{Q} = \mathcal{Q}^{(i)}$*,* $\mathcal{P} = \mathcal{Q}^{(i+1)}$ *for* $i \in \{0, 1, 2\}$ *and* $q \in V\left(Q\right)$*,* $p \in V\left(P\right)$*.*

3. *The root is the unique null assignment* $\mathcal{N}a := (n, n')$*.*

This definition has a self-referential flavour: the validity of quotient assignments (5.7) depends on the existence of a more global quotient morphism, which itself is constructed by concatenation of quotient assignments. This explains why several passes are necessary to construct $T_{\mathcal{Q}a}$, as explained in Sec. 5.2.3 to 5.2.5.

Quotient Morphisms Tree

A third tree can be built for the complete quotient morphisms. Let $\mathcal{P}\phi_2 : P \to P'$ and $\mathcal{Q}\phi_1 : Q \to Q'$ be two quotient morphisms. Their inclusion derives from the quotient component inclusions in a natural way, i.e. one quotient morphism is included in another if all their single quotient assignments follow the same inclusion structure.

$$\mathcal{Q}\phi_1 \subseteq \mathcal{P}\phi_2 :\Leftrightarrow \mathcal{Q}\phi_1\left(q\right) \subseteq \mathcal{P}\phi_2\left(p\right) \quad \forall q, p : q \subseteq p \tag{5.10}$$

Definition 43. *The* ***quotient morphism tree*** $T_{\mathcal{Q}\phi}$ *between a pair of strict digraphs* D *and* D' *is a rooted, directed tree.*

1. *The Vertex set* $V\left(T_{\mathcal{Q}\phi}\right)$ *will contain all quotient morphisms*

$$\mathcal{Q}\phi : Q \to Q'. \tag{5.11}$$

2. *The arrow set connects each pair of successive quotient morphisms compatible with inclusion*

$$(\mathcal{P}\phi_2, \mathcal{Q}\phi_1) \in A\left(T_{\mathcal{Q}\phi}\right) \Leftrightarrow \mathcal{Q}\phi_1 \subseteq \mathcal{P}\phi_2 \tag{5.12}$$

and belonging to successive quotient levels

$$\begin{cases} \mathcal{P} := \mathcal{Q}^{(i)} \\ \mathcal{Q} := \mathcal{Q}^{(i+1)} \end{cases} \tag{5.13}$$

for $i \in \{0, 1, 2\}$ *respectively.*

3. *The root is the unique morphism* $\mathcal{N}\phi$ *connecting the two unique null quotient components* $n \mapsto n'$*.*

5.2 Quotient Morphisms Enumeration

Quotient morphisms can be enumerated from the broadest null level to the finest full level. It will be the task of the *BollyMorpher* rubette described in Sec. 7.2.6 to help the user explore such a list and compose a complete mapping of contrapuntal intervals in a structured way. This procedure is described step by step below. The first phases presented in steps 5.2.1 to 5.2.4 constitute the downwards exploration of the quotient structure. This preliminary task does not grow as fast as does the inspection of all possible mappings, and provides the information necessary to prune the search tree early. The last two steps in Sec. 5.2.5 consist in assembling the quotient morphism tree from the lower levels back to the null level.

Inputs

A pair of counterpoint worlds. Both the source counterpoint world CW and the target counterpoint world CW' may be local or global, and possibly the same if one is interested in automorphisms.

Output

The quotient morphism tree $T_{\mathcal{Q}\phi}$ containing all possible strict digraph morphisms between the two worlds, organised after their quotient hierarchy. Note that this tree may be empty if no morphism can be constructed between the two worlds.

Steps

The following road map summarises the strategy of the algorithm. Each step will be explained in its own section below:

1. Construction of the Quotient Graphs
2. Hierarchical organisation of the quotient components
3. Hierarchical organisation of individual quotient assignments
4. Enumeration of local combinations of child quotient assignments
5. Enumeration of complete quotient mappings

Example 15. *To illustrate the procedure, we will transform the example counterpoint p shown in Fig. 5.1 from the Fuxian world (Δ_{87}, or 82 in* ToM*)*

$$K = \{[0], [3], [4], [7], [8], [9]\}$$

into the major dichotomy's world Dur *(Δ_{64})*

$$K' = \{[2], [4], [5], [7], [9], [11]\}$$

which is connected to the indian mela scales.[1] *Each step of the algorithm will be illustrated with an application to this example, that will be used through the whole chapter.*

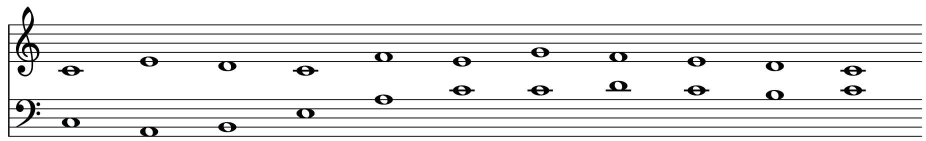


Figure 5.1: Score of the original (source) Fuxian counterpoint p. Cantus firmus on the first staff, discantus below.

5.2.1 Build the Quotient Graphs

For each of the source and target worlds, compute the strict digraphs

$$D := \mathcal{D}CW \quad D' := \mathcal{D}CW' \tag{5.14}$$

along with the four quotient digraphs.

$$\begin{array}{ccc} i & CW & CW' \\ \hline 0 & N := \mathcal{N}D & N' := \mathcal{N}D' \\ 1 & W := \mathcal{W}D & W' := \mathcal{W}D' \\ 2 & H := H & H' := H' \\ 3 & F := \mathcal{F}D & F' := \mathcal{F}D' \end{array} \tag{5.15}$$

We can save a lot of computational efforts if we use the fact that, in the special case of strict digraphs generated by counterpoint worlds, a finer quotient component is always entirely contained in a broader component, as was explained in Sec. 4.3. The information gathered at previous levels can be reused, so that it is not necessary to start from scratch again each time we reach a new level. Start with the weak components, and compute the strong components inside each of them. Linear-time algorithms exist for both operations, and can be found in [CLRS09]. However, for building homogeneous components, the computation time may grow in $O\left(n^4\right)$ if we check all pairs of contrapuntal intervals for their neighbourhoods. Since we know that these components are contained inside strong components, it suffices to verify pairs belonging to a same strong component.

Example 16. *The quotient homogeneous graphs H and H' of the source world CW respectively the target world CW' are shown in Fig. 5.2.*

[1] See page 658 in Sec. 31.4.2 of the *Topos of Music*.

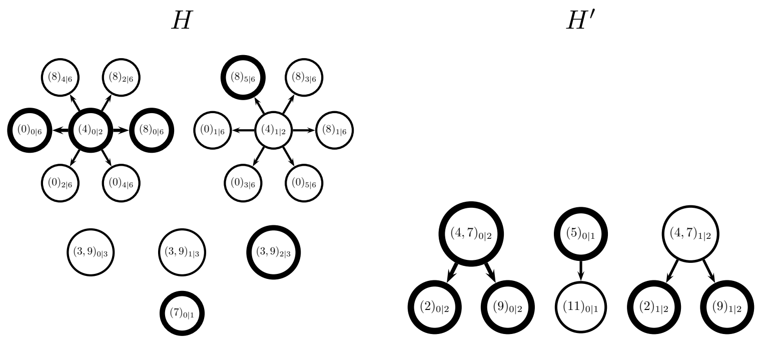


Figure 5.2: Homogeneous quotient graphs of the Fuxian (left) and Dur (right) worlds. Thick lines denote components containing contrapuntal intervals used by the example counterpoint p of Fig. 5.1 in the Fux world, and their image under the mapping into the Dur world proposed across all examples of this chapter.

5.2.2 Build the Quotient Component Tree

Once all quotient components—the vertices of the quotient component graphs—have been computed, they can be organised into the tree structures $T_{\mathcal{Q}D}$ and $T_{\mathcal{Q}D'}$ described in Sec. 5.1. This can be done by walking down the hierarchy recursively.

Initialisation

First, create an empty tree $T_{\mathcal{Q}D}$ and add the unique null quotient component $n \in V(N)$ as a first vertex. This will be the root of $T_{\mathcal{Q}D}$.

Recursion

Let $p \in V(P)$ be a given quotient component. Loop through the child quotient components q of the next level, adding every component satisfying the inclusion relation, see Def. 37.

$$(p, q) \in A(T_{\mathcal{Q}D}) \Leftrightarrow q \subseteq p \quad \forall q \in V(Q) \tag{5.16}$$

Apply this procedure again on each added child component as long as a further quotient level exists.

Termination

The procedure ends when the bottom of the quotient hierarchy is reached, i.e. theoretically at the full level ($i = 3$). In practice, we build the tree only until the homogeneous level ($i = 2$), which entirely describes the structure of a counterpoint world. The final full level leaves can be easily computed on the fly, once the homogeneous structure is known, without bloating the quotient component tree.

Repeat the same procedure for the target strict digraph D', and generate the target quotient graph tree $T_{\mathcal{Q}D'}$.

Example 17. *The global (regardless of the particular counterpoint being transformed) quotient component tree $T_{\mathcal{Q}D'}$ of the Dur world splits into three branches at the weak level, each one corresponding to a distinct weak component w'. Two of them are isomorphic, see Fig. 5.3.*

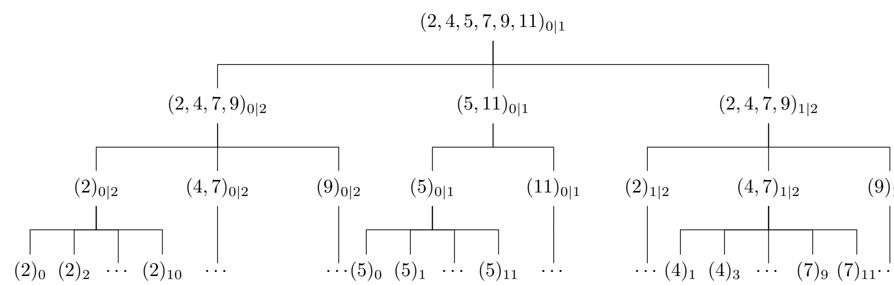


Figure 5.3: The Quotient component tree $T_{\mathcal{Q}D'}$ of the major dichotomy Δ' (Dur). Displayed at the top, the single null component n serves as a root. Intermediate levels are weak and homogeneous. For readability, not all leaves, the full components f, are represented.

5.2.3 Build the Quotient Assignment Tree

The quotient assignment tree $T_{\mathcal{Q}a}$ will be constructed in three steps. A first pass will add assignments sequentially, level after level.

One difficulty arises from the fact that not every quotient assignment $q \mapsto q'$ is part of a valid quotient morphism $\mathcal{Q}\phi$. A preliminary filtering can be done to reduce the number of candidates. There are a few general considerations, described below, that allow us to discard assignments we know in advance to be useless. They apply at the weak and homogeneous levels, and follow from the injectivity of homogeneous quotient morphisms, as explained in Sec. 4.3.2. Such criteria, while easily computable, provide only a necessary, not a sufficient condition for the assignment to be part of a strict digraph morphism, as expressed in (5.5). Two supplementary operations of filtering will be necessary to clean up the tree before it reaches its definitive state. They will be explained in the next section.

Weak Component Embedding Condition

Weak quotient morphisms $\mathcal{W}\phi$ need not to be injective. Nevertheless, since homogeneous morphisms $\mathcal{H}\phi$ have to be injective, it makes no sense to map a weak component w into a target component w' if the underlying homogeneous digraph $H[w]$ won't fit into the corresponding homogeneous digraph $H'[w']$. A simple way to guarantee that there is enough space in w' for w is to inspect the graph parameters:

$$\mathbb{1}_{\mathcal{Q}a}\left((w,w')\right) = \top \Rightarrow \begin{cases} n\left(H[w]\right) \leq n\left(H'[w']\right) \\ a\left(H[w]\right) \leq a\left(H'[w']\right) \end{cases} \tag{5.17}$$

Every weak quotient assignment not fulfilling this condition can be discarded from the quotient assignment tree $T_{\mathcal{Q}a}$.

Homogeneous Neighbourhood Embedding Condition

At the homogeneous level, injectivity follows from the preservation of connection profiles. This property holds not only for the mapping of vertices between homogeneous graphs, but also for the mapping of arrows $(h_0, h_1) \in A(H)$. The degrees of vertices can be used to asses the ability of a target component h' and its surroundings to welcome the mapping of a source component h. We simply have to compare the in-degrees (number of arrows pointing to) $a^{(0)}$ and out-degrees (number of arrows anchored at) $a^{(1)}$, as seen in Fig. 5.4.

$$\mathbb{1}_{\mathcal{Q}a}((h, h')) = \top \Rightarrow \begin{cases} a_H^{(0)}(h) \leq a_{H'}^{(0)}(h') \\ a_H^{(1)}(h) \leq a_{H'}^{(1)}(h') \end{cases} \tag{5.18}$$

Add only homogeneous quotient assignments that satisfy this condition.

Homogeneous Complement Embedding Condition

The previous criterion sets a lower bound on the in- and out-degrees of the target component h', but no upper bound. For example, nothing prevents a component h having only a few neighbours from being assigned to a component h' having many of them. This would turn a lot of components of the target's neighbourhood $N_{H'}(h')$ into components unavailable for mapping further vertices that are not directly connected to h. Here, the neighbourhood has to be understood in the *weak* sense: considering every in-coming and out-going arrow. Since the procedure tries to squeeze the entire source homogeneous digraph H into the target homogeneous digraph H', we have to keep as most components available for further mapping as we can. This criterion ensures that a particular assignment leaves enough space for the remaining components.

$$\mathbb{1}_{\mathcal{Q}a}((h, h')) = \top \Rightarrow \begin{cases} n(H \backslash N_H(h)) \leq n(H' \backslash N_{H'}(h')) \\ a(H \backslash N_H(h)) \leq a(H' \backslash N_{H'}(h')) \end{cases} \tag{5.19}$$

Add only homogeneous quotient assignments that satisfy this condition.

At this stage, the quotient assignment tree $T_{\mathcal{Q}a}$ may still contain too many quotient assignments, because the three previous conditions do not provide a sufficiently fine filtering.

Note that every full assignment is allowed as long as it is a child of a valid homogeneous assignment.

Example 18. *Fig. 5.5 shows how major thirds can be mapped into major thirds or fifths.*

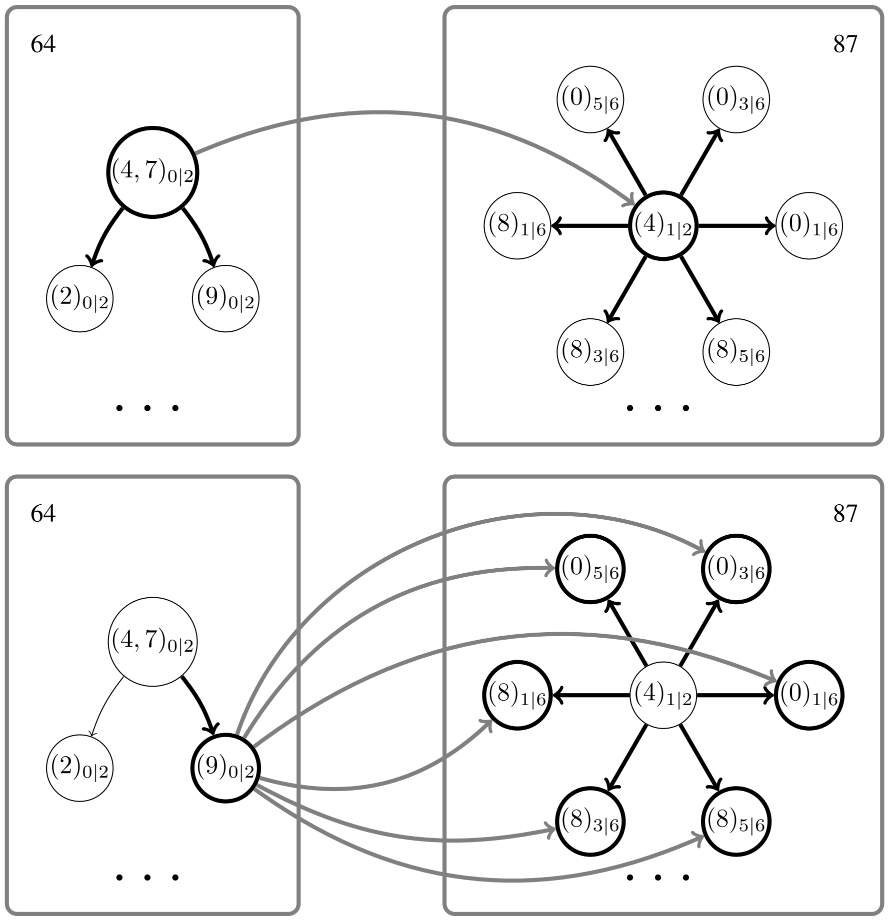


Figure 5.4: A target homogeneous component must be surrounded by enough incoming and out-going arrows in order to accept a mapping. Possible mappings between the 2-branched star of the Dur world Δ_{64} to the 6-branched star of the Fuxian world Δ_{87} (82 in *ToM*) are displayed with grey arrows.

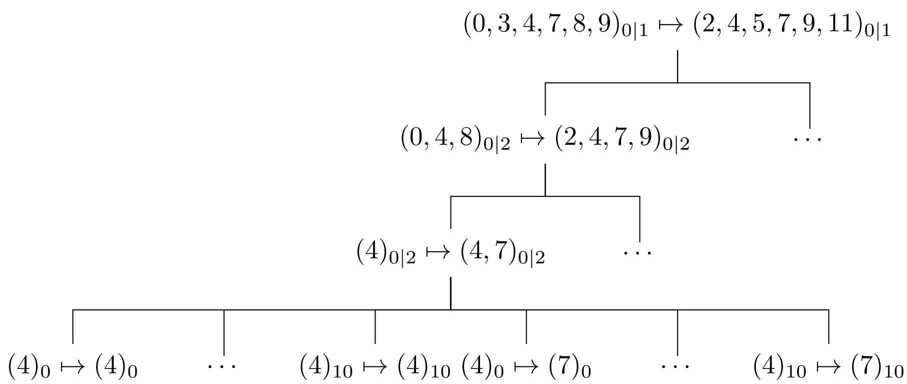


Figure 5.5: A branch of the quotient assignment tree $T_{\mathcal{Q}a}$ generated by the example counterpoint p, Fux and Dur worlds.

5.2.4 Enumerate Local Child Quotient Assignment Combinations

We now turn to the *sufficient* aspect of condition (5.8), which implies further filtering of the quotient assignments. Once single assignments have been selected and organised into a tree, we verify that each of them indeed yields valid combinations of subquotient assignments, or child assignments, i.e. that they allow proper *local* mappings. We assign to each quotient assignment (p, p') a set of local combinations of child quotient assignments.

$$\begin{aligned} V\left(T_{\mathcal{Q}a}\right) &\longrightarrow \mathscr{P}\left(V\left(T_{\mathcal{Q}a}\right)\right) \\ (p, p') &\mapsto \mathcal{Q}\Phi_{(p,p')} \end{aligned} \tag{5.20}$$

where $\mathcal{Q}\Phi_{(p,p')}$ designates the set of local quotient morphisms, i.e. mappings restricted to assign subquotient components of p and p' only. Global assignments involving the complete vertex set $V\left(P\right)$ will not show up before the last stage. We enumerate all possible combinations of child assignments

$$\{(q_1, q_1'), \dots, (q_I, q_I')\} \subseteq N^{(1)}_{T_{\mathcal{Q}a}}\left((p, p')\right) \tag{5.21}$$

yielding a valid local quotient morphism, which means that the three following conditions have to be satisfied:

1. Exactly one child assignment (q, q') must be attached to every child component q of p.

$$\begin{aligned} I &= \left|N^{(1)}_{T_{\mathcal{Q}a}}\left((p, p')\right)\right| \\ \{q_1, \dots, q_I\} &= N^{(1)}_{T_{\mathcal{Q}a}}\left((p, p')\right) \end{aligned} \tag{5.22}$$

   If this is not the case, do not add the combination to $\mathcal{Q}\Phi_{(p,p')}$.

2. At the homogeneous level, the connectivity structure must be preserved. Let $p = w$ and $p' = w'$, the q_is become h_is in (5.21).

$$(h_i, h_j) \in A(H) \Leftrightarrow (h'_i, h'_j) \in A(H') \quad \forall i, j \in \{1, \ldots, I\} \tag{5.23}$$

   If this is not the case, do not add the combination to $\mathcal{Q}\Phi_{(p,p')}$.

3. At the homogeneous level the set must also be injective.

$$h_i \neq h_j \Rightarrow h'_i \neq h'_j, \quad \forall i, j \in \{1, \ldots, I\} \tag{5.24}$$

   If this is not the case, do not add the combination to $\mathcal{Q}\Phi_{(p,p')}$.

Start with an empty set $\mathcal{Q}\Phi_{(p,p')} := \varnothing$. Enumerate all combinations of child quotient assignments (5.21) and inspect each one in turn. If all three conditions are met, the subquotient assignment combination $\{(q_1, q'_1), \ldots, (q_I, q'_I)\}$ defines a valid local subquotient mapping and can be added to $\mathcal{Q}\Phi_{(p,p')}$. At the end, if $\mathcal{Q}\Phi_{(p,p')}$ is not empty, (p, p') can be left in the quotient assignment tree $T_{\mathcal{Q}a}$.

On the contrary, every assignment which is not a leaf (a full assignment) and yields an empty set $\mathcal{Q}\Phi_{(p,p')}$ is invalid. It constitutes a dead-end that prevents us from reaching the final full-level, from which a real counterpoint world morphism can eventually be built. It has to be removed from the quotient assignment tree $T_{\mathcal{Q}a}$.

$$\mathcal{Q}\Phi_{(p,p')} = \varnothing \Rightarrow \mathbb{1}_{\mathcal{Q}a}((p, p')) = \perp \tag{5.25}$$

Example 19. *The weak assignment $\mathcal{W}a = ((0, 4, 8)_{1|2}, (2, 4, 7, 9)_{1|2})$ yields three possible homogeneous assignments:*

$$\begin{aligned} \mathcal{H}a_1 &= ((8)_{5|6}, (2)_{1|2}) \\ \mathcal{H}a_2 &= ((8)_{5|6}, (4, 7)_{1|2}) \\ \mathcal{H}a_3 &= ((8)_{5|6}, (9)_{1|2}) \end{aligned} \tag{5.26}$$

The first and last ones are valid, but the second assignment does not fulfil the neighbourhood embedding condition (5.18): Using the central node $(4, 7)_{1|2}$ (see Fig. 5.2), we discard two neighbour positions. Because of the requirement for connectivity and neighbourhood preservation, there is not enough space left for mapping the rest of the singletons.

Discarding a quotient assignment (q, q') from $T_{\mathcal{Q}a}$ may have consequences at a higher level. Local child mappings in the ancestor chain may have become invalid if they involve the removed assignment, so some additional verifications and possible cleaning are necessary.

1. Get the parent quotient assignment.

$$(p, p') := N^{(0)}_{T_{\mathcal{Q}a}}((q, q')) \tag{5.27}$$

2. Discard every child assignment combination of the parent's assignment in which (q, q') is involved.

$$\mathcal{Q}\Phi_{(p,p')} := \Big\{ \{(q_1, q_1'), \ldots, (q_I, q_I')\} \in \mathcal{Q}\Phi_{(p,p')} \Big| \\ (q_i, q_i') \neq (q, q'), \forall i \in \{1, \ldots, I\} \Big\} \quad (5.28)$$

3. If $\mathcal{Q}\Phi_{(p,p')} = \varnothing$ after step 2, remove also the parent assignment (p, p') from $T_{\mathcal{Q}a}$ and go step 1.

4. Stop climbing up the ancestor hierarchy either if valid local combinations still exist ($\mathcal{Q}\Phi_{(p,p')} \neq \varnothing$), or if the root (null quotient level) has been reached: $(q, q') = (n, n')$.

The quotient assignment tree $T_{\mathcal{Q}a}$ could have been emptied by this filtering procedure. If this happens, no morphisms can be constructed between the two counterpoint worlds. Stop and return $T_{\mathcal{Q}\phi} := \varnothing$.

5.2.5 Enumerate Global Quotient Assignment Combinations

Global combinations, that finally define a complete quotient morphism $\mathcal{Q}\phi$, are obtained from concatenations of local combinations. The set of all quotient morphisms at level $\mathcal{Q}$ is defined as:

$$\mathcal{Q}\Phi := \cup_{\mathcal{P}\phi \in \mathcal{P}\Phi} \uplus_{p' = \mathcal{P}\phi(p)} \mathcal{Q}\Phi_{(p,p')} \quad (5.29)$$

It can be computed once the set one level above, $\mathcal{P}\Phi$ is known. The recursion begins with, the null level, which is trivially a singleton: $\mathcal{N}\Phi = \{\mathcal{N}\phi\}$.

Example 20. *Fig. 5.6 shows a part of the branch of the quotient morphism tree $T_{\mathcal{Q}\phi}$ chosen for this series of examples. It contains the individual quotient assignment shown in Fig. 5.5 and leads to the morphism proposed in Tab. 5.1.*

This operation needs a supplementary verification at the homogeneous quotient level. There, we have to make sure that an induced homogeneous mapping will indeed be injective, in case its parent weak mapping is not. If two different weak components are mapped into a single one, the two images must be kept separate, i.e. not overlap (injectivity) nor be connected (preservation of the connectivity structure). Let $\{w_i\}_{i=1}^{I}$ be a set of source weak components, all mapped into the same target weak component w'. We extend the notion of local mappings between single components to the case of more than one source weak component:

$$\mathcal{H}\Phi_{(\{w_i\},w')} := \Big\{ \mathcal{H}\tilde{\phi} : H[\cup w_i] \to H[w'] \Big| \\ \exists \mathcal{H}\phi : H \to H' : \mathcal{H}\tilde{\phi} = \mathcal{H}\phi|_{\{w_i\}} \Big\}. \quad (5.30)$$

The set of weak quotient morphisms will be constructed by concatenation of combinations of multiple source components. The simple case is described by $I = 1$.

$$\mathcal{H}\Phi := \cup_{\mathcal{W}\phi \in \mathcal{W}\Phi} \uplus_{w' = \mathcal{W}\phi(w)} \mathcal{H}\Phi_{\{w_i\},w'} \quad (5.31)$$

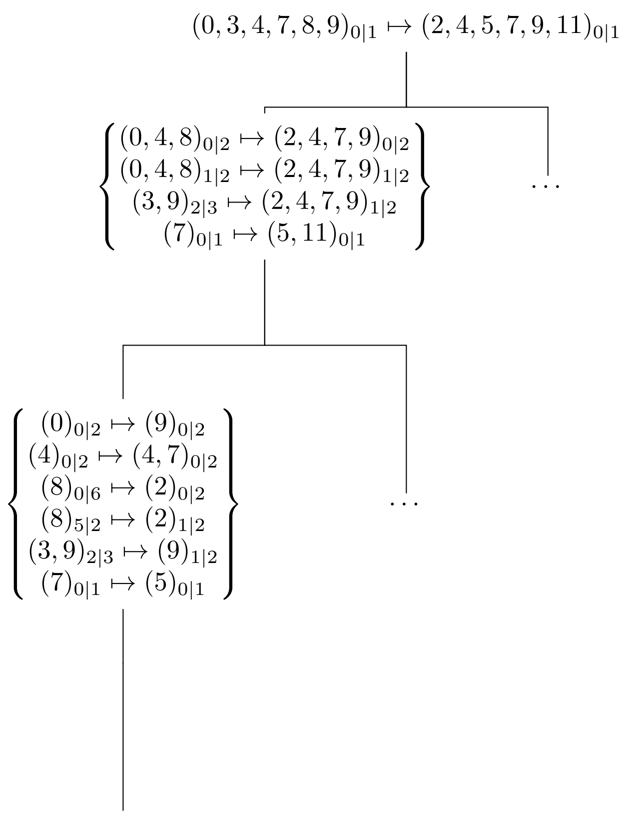


Figure 5.6: The beginning of a branch of the huge quotient morphism tree $T_{\mathcal{Q}\phi}$ for mapping the Fuxian into the major dichotomy world.

Example 21. *Both weak components* $(0,4,8)_{1|2}$ *and* $(3,9)_{2|3}$ *can be mapped simultaneously into the single weak component* $(2,4,7,9)_{1|2}$, *and still lead to two injective local homogeneous morphisms:*

$$\mathcal{H}\Phi_{\left(\{(0,4,8)_{1|2},(3,9)_{2|3}\},(2,4,7,9)_{1|2}\right)} = \left\{ \left\{ \begin{array}{c} ((8)_{5|6} \mapsto (2)_{1|2}), \\ ((3,9)_{2|6} \mapsto (9)_{1|2}) \end{array} \right\}, \left\{ \begin{array}{c} ((8)_{5|6} \mapsto (9)_{1|2}), \\ ((3,9)_{2|6} \mapsto (2)_{1|2}) \end{array} \right\} \right\}$$

At the null and full levels, the concatenation of local combinations into a complete quotient morphism is straightforward.

Example 22. *A proposition for one final mapping of contrapuntal consonances is shown in Tab. 5.1. Many other choices are possible.*

Table 5.1: Mapping contrapuntal intervals ξ and homogeneous components h of the example counterpoint p from the Fuxian world CW into the Dur world CW'.

	CW	CW'	
ξ	h	h'	ξ'
0_0	$(0)_{0\|6}$	$(9)_{0\|2}$	9_0
3_2	$(3,9)_{2\|3}$	$(9)_{1\|2}$	9_3
3_5			9_5
4_4	$(4)_{0\|2}$	$(4,7)_{0\|2}$	7_4
7_4	$(7)_{0\|1}$	$(5)_{0\|1}$	5_4
7_7			5_7
8_5	$(8)_{5\|6}$	$(2)_{1\|2}$	2_5
8_0	$(8)_{0\|6}$	$(2)_{0\|2}$	2_0

5.3 Counterpoint World Morphisms Enumeration

A mapping rule for the contrapuntal intervals, while acting on the counterpoint structure, will not provide a transformation rule for *real* music. Some supplementary work is necessary for mapping real pitches, not classes.

5.3.1 From Notes to Counterpoints

The transformation of pitches into pitch classes is straightforward. A reference pitch x_0 determining which is the zero class should usually be provided separately. But remember that the rules of counterpoint are invariant under a global transposition: it does not matter where the reference pitch lies.

Let x be the cantus firmus pitch and z the discantus pitch. The construction of an oriented contrapuntal intervals is described in Sec. 2.2.1, in particular the orientation which is given by (2.47):

$$\hat{\zeta} := ([x - x_0]_n + \varepsilon.[|z - x|]_n, \Omega_{\mathrm{sgn}(z-x)}). \tag{5.32}$$

The eleven steps of the example counterpoint p need to be encoded before they can be used. See the following example for an illustration.

Example 23. *Take the counterpoint depicted in Fig. 5.1. It leads to the following numerical values: for each step s, the MIDI pitches of the cantus firmus x_s and the discantus z_s lead to the pitch class of the cantus firmus $[x_s]$, the interval class $[y_s]$ and the orientation Ω_s, given a reference pitch $x_0 = 60$ and a division of the octave into $n = 12$ semitones.*

s	x_s	z_s	$[x_s]$	$[y_s]$	Ω_s
0	60	48	[0]	[0]	$-$
1	64	45	[4]	[7]	$-$
2	62	47	[2]	[3]	$-$
3	60	52	[0]	[8]	$-$
4	65	57	[5]	[8]	$-$
5	64	60	[4]	[4]	$-$
6	67	60	[7]	[7]	$-$
7	65	62	[5]	[3]	$-$
8	64	62	[4]	[4]	$-$
9	62	59	[2]	[3]	$-$
10	60	60	[0]	[0]	$-$

5.3.2 Morphing Counterpoints

Strict digraphs contain only contrapuntal consonances. If we encounter a dissonance, we have to use the polarity relations (3.14) to map it correctly.

$$\zeta' := \begin{cases} \phi(\zeta) & \text{if } \kappa(\zeta) = \top \\ p^{\bullet}_{\Delta'} \circ \phi \circ p^{\bullet}_{\Delta}(\zeta) & \text{otherwise} \end{cases} \tag{5.33}$$

Example 24. *The example counterpoint p follows the rules of the Fuxian world and contains no dissonances, so that formula (5.33) does not need to be applied. Suppose we had a major second in our counterpoint, say 2_0. The source autocomplementary function $p_\Delta = e^{[2]}.[5]$ maps it to its dual consonance, namely the octave 0_0. Tab. 5.1 tells that this contrapuntal interval then gets mapped into the consonance 9_0. We finally apply the target autocomplementary function $p_{\Delta'} = e^{[11]}.[11]$ to find the target dissonance: 8_0.*

5.3.3 From Counterpoints to Notes

Information that has been lost at the first stage has to be provided again, or replaced. In order to transform classes back into real pitches, we have to specify the octave in which the note will eventually lie. Any choice is suitable: one may prefer the original octave, or try to minimize jumps between notes in the melodies. A reference pitch x'_0 that will fix a base note common to the whole counterpoint must also be given. It acts as a global translation parameter and does not affect the counterpoint rules. For every contrapuntal interval $\zeta' = x' + \varepsilon.y'$ we can reconstruct a cantus firmus pitch $\tilde{x}'$ and a discantus pitch $\tilde{z}'$ in the following way:

$$\tilde{x}' := x'_0 + k \cdot n + \mathrm{d}_{\mathcal{C}_{n'}}(x') \tag{5.34}$$

$$\tilde{z}' := \tilde{x}' + o \cdot (\mathrm{d}_{\mathcal{C}_{n'}}(y') + l \cdot n') \tag{5.35}$$

where the orientation $o \in \{-1, +1\}$, and the octave indices $k \in \mathbb{Z}$ and $l \in \mathbb{N}$ must be specified.

Nothing prevents the chromatic gamut sizes n and n' from being different. This is a nice by-product of the abstraction implied by graphs: homomorphisms care only about connections, and not about the particular nature of the vertices.

A complete musical transformation is in fact the composition of two successive transformations: A strict digraph morphism ϕ responsible for carrying the contrapuntal structure, together with a set of imputations for instantiating the more abstract contrapuntal intervals in the real world of pitches.

Example 25. *One choice (among many) for real pitches of a transformed counterpoint is shown in Fig. 5.7. They are derived from the pitch and interval classes shown in the right most column of Tab. 5.1.*

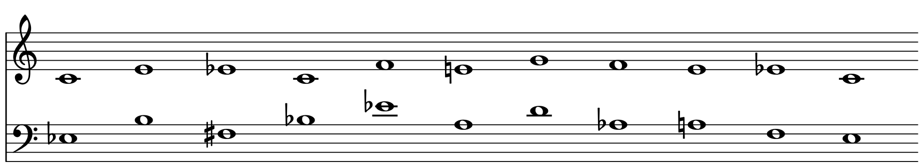


Figure 5.7: The Fuxian counterpoint p mapped into the major dichotomy world CW'. Pitches have been moved to accommodate with the new compositional rules. Many other choices would be possible.

5.4 Global Counterpoint World Morphisms

As mentioned earlier, it is not always possible to construct a strict digraph morphism ϕ between two given strict digraphs D and D'. A simple verification can be made by applying the weak component embedding condition (5.17) to the whole homogeneous graphs H and H':

$$\exists \phi : D \to D' \Rightarrow \begin{cases} n(H) \leq n(H') \\ a(H) \leq a(H') \end{cases} \tag{5.36}$$

The necessary but not sufficient character of this requirement does not guarantee the existence of a morphism. Nevertheless, the algorithm presented in this chapter is quite demanding in terms of computations, and this criterion provides a quick and easy way to avoid useless work in situations that can be pointed out by (5.36).

With an algorithm like the one presented in this chapter, we are now able to answer one question that motivated this research: is it possible to morph the European Fuxian world into the Indian Dur world ? We saw it can be achieved locally, at least for the counterpoint p shown in the examples. But the global embedding condition (5.36) tells us that there are too many vertices and arrows in the source homogenous digraphs to do it globally:

$$\begin{aligned} 18 = n\left(H\right) > n\left(H'\right) = 8 \\ 40 = a\left(H\right) > a\left(H'\right) = 13 \end{aligned} \tag{5.37}$$

The same criteria applies for any of the five other worlds in $\mathcal{C}_{12}$, which means that it is impossible to leave the Fuxian world with a global morphism. There is not enough space for placing all separate components of the Fuxian world's homogeneous digraph into any other homogeneous digraph. This can be seen from the homogeneous digraphs listed in Fig. 4.15. In the opposite direction, the Fuxian world contains too many singletons, i.e. it lacks arrows, so that the arrows of other worlds can not be embedded into it. The Fuxian world is not a special case: two other worlds, namely Δ_{72} and Δ_{82}, are also isolated. They are not related by a global morphism to any other world in $\mathcal{C}_{12}$, as shown in Fig. 5.8. Only local transformations of counterpoints can be performed, see Fig. 5.9. Note that the procedure may not work for all counterpoints: only those whose set of relevant contrapuntal intervals can be embedded into a global world will do.

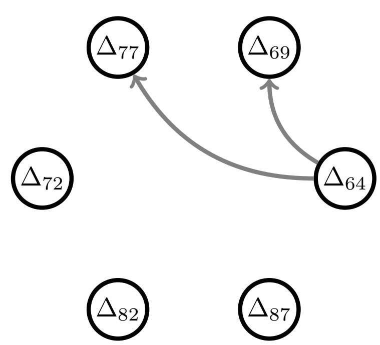


Figure 5.8: Global counterpoint world morphisms in $\mathcal{C}_{12}$. Arrows indicate the existence of global morphisms between isometry classes of counterpoint worlds. As far as global morphisms are concerned, the classes 72, 82 and 87, which represents the Fuxian world, are completely separated from the other five worlds. This digraph is reflexive. For readability, loops (i.e. global automorphisms) are not shown. Class numbers correspond to 64, 69, 71, 75, 78 respectively 82 in *ToM*.

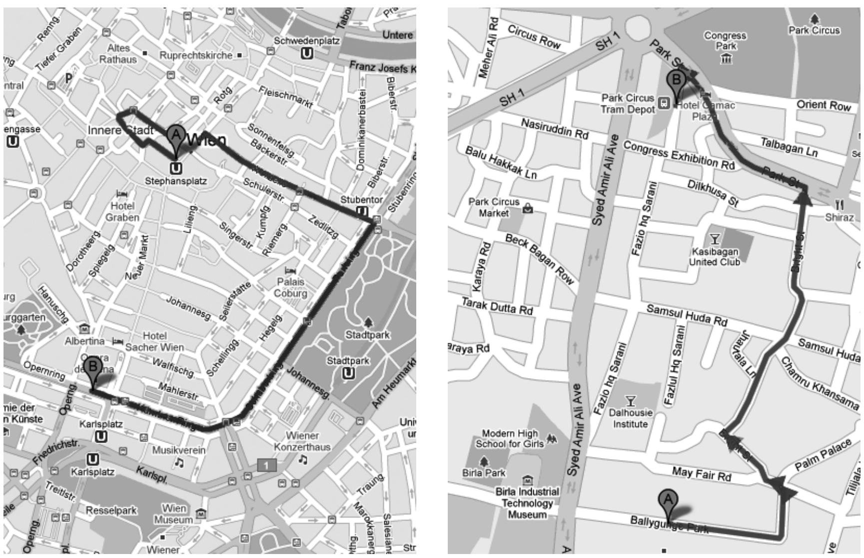


Figure 5.9: If we go back to the traffic analogy in Fig. 4.16, transforming a global world into another is like sending a whole city map onto another one. A local morphism is far less demanding: only a given travel through a city is mapped to a new travel. This results in mapping only small parts of the entire city maps (examples taken from *Google™maps*: `http://maps.google.com/`).

5.5 Complexity

A last word about complexity and the amount of computational task involved in the algorithm presented in this chapter. It does not so much depend on the global order n and size a of the homogeneous graphs H and H', than on their local connectivity structure.

The filters described in Sec. 5.2.3 and the pruning of the search tree will work effectively only under the presence of subgraphs that are hard to embed in the target digraph.

The trick is to keep the quantity of local subcomponents assignments low. A counter-example is given by the automorphisms of the Fuxian world Δ_{87} (82 in *ToM*). Fig. 4.15 shows six weak components: two stars and four singletons at the homogeneous level. Singletons need to be mapped to singletons, so there are $4! = 24$ possible local homogeneous mappings, a fairly low number. While there are only $2! = 2$ possibilities to map the stars, the problem arises from the many possibilities to map the branches: six identical configurations in each leading to $6! = 720$ different combinations of individual assignments. This too may seem quite reasonable at a first sight, but putting it all together, their global combination yields a far higher value, namely

$$2! \cdot 6! \cdot 6! \cdot 4! = 24883200 \tag{5.38}$$

homogeneous automorphisms in the Fuxian world. Compare this number with the $2! \cdot 2! \cdot 2! = 8$ homogeneous automorphims in the DUR world (resulting from the mapping of two stars with two branches each). It is really important to limit the size of the orbits of homogeneous components under all possible automorphisms, i.e. that each vertex be as specific as possible. The branches of the two stars explanation why the Fuxian world appears with the longest computation time in Tab. 5.2. This table also shows that the global embedding condition (5.36) works in most situations, but fails to predict the lack of embedding in at least three cases, letting the algorithm spend a lot of time in the search tree before it propagates incompatibilities back to the null level and empties the quotient assignment tree.

The situation for local morphisms is quite similar to the Fuxian world, because many counterpoints will produce an embedded subgraph often populated with a majority of singletons. They form the easiest subgraphs to embed, and this again will produce large orbits, with many combinations of quotient component assignments. There will always be some kind of trade-off between a counterpoint too short to allow a reasonable quantity of embeddings to be enumerated, and a counterpoint too long to be embedded at all.

5.6 Summary

The reader may wonder why a complete exploration of all morphisms is necessary, if the user is interested in only a particular morphism. We mentioned earlier how this task may need a lot of computation time and memory space.

The reason behind this exhaustive enumeration is the implementation described in Chap 7, which is intended to be *user-friendly*: It should be possible to choose

Table 5.2: Execution times for the enumeration of global world morphisms, in seconds. These values only give orders of magnitudes, since the computational time may also have served for garbage collection peculiar to the Java virtual machine, or other processes. It may also vary, depending on computer and Java version installed. Rows correspond to source worlds, columns to target worlds, automorphisms are located on the diagonal. Note that the table is not symmetric: when a world can be embedded into another, the converse is not necessarily true. Bold values correspond to combinations of worlds for which global morphisms really exist, as depicted in Fig. 5.8. Class numbers correspond to 64, 69, 71, 75, 78 respectively 82 in *ToM*.

	Δ'					
Δ	64	69	72	77	82	87
64	**2**	**1**	1	**48**	19	2
69	0	**1**	0	337	322	6
72	0	0	**1**	1	2	0
77	0	1	0	**6**	0	1
82	1	0	0	0	**2**	0
87	0	0	0	158	0	**397**

freely among possible assignments of quotient vertices and contrapuntal consonances, in such a way that the user can build a complete morphism step-by-step, without getting trapped in invalid combinations. In order to achieve this, we need to construct the complete quotient assignment tree. But the global knowledge of all assignments and their combinations is necessary to build it, even if we are working at the local level of a single isolated assignment. This is due to the *recursive* character of Def. 42.

The enumeration algorithm follows a simple backtracking strategy. This brute-force method works because the hierarchical organisation of strict digraphs into quotient graphs allows us to detect invalid mappings, and prune the search tree early enough. The main task of the algorithm consists in constructing the quotient assignment tree, which is not so much done by adding valid assignments, than by deleting invalid ones. The filtering procedure occurs in three steps, by traversing the quotient hierarchy, as shown in Fig. 5.10:

1. Horizontally: The three embedding conditions (5.17), (5.18), and (5.19) allow some preliminary filtering.

2. Downwards: sec. 4.3.2 describes how broader assignments guide finer ones in the hierarchy. We propagate from the top (broadest) to the bottom (finest) level, filtering out every assignment that is incompatible with higher levels.

3. Upwards: One task we cannot avoid any longer is to verify if the individual assignments, once combined into a complete morphism, preserve the connectivity structure of the graphs. The algorithm aims to restrict this computationally intensive $O\left(n^2\right)$ operation to the smallest possible subgraphs, so that we can avoid the cost of a check performed on the entire graph. In this last step, we climb up the hierarchy, verifying if the local children of homogeneous and then weak

components allow valid (structure preserving) local combinations, as described in Sec. 5.2.4.

The already mentioned recursive character of Def. 42 has consequences on this last phase too: several passes may be necessary, since the detection of an invalid assignment (due to the absence of valid children combinations) may invalidate all combinations of siblings involving this particular assignment. The parent assignment may then become invalid too if all of its children combinations include the invalid assignment deleted before. Deletions of assignments may thus be propagated up to the root. This iterative process stops either once the tree is completely empty, or once it is stabilised: i.e. all local combinations and hence parent assignments are valid at every level, so that it has not to be altered any more.

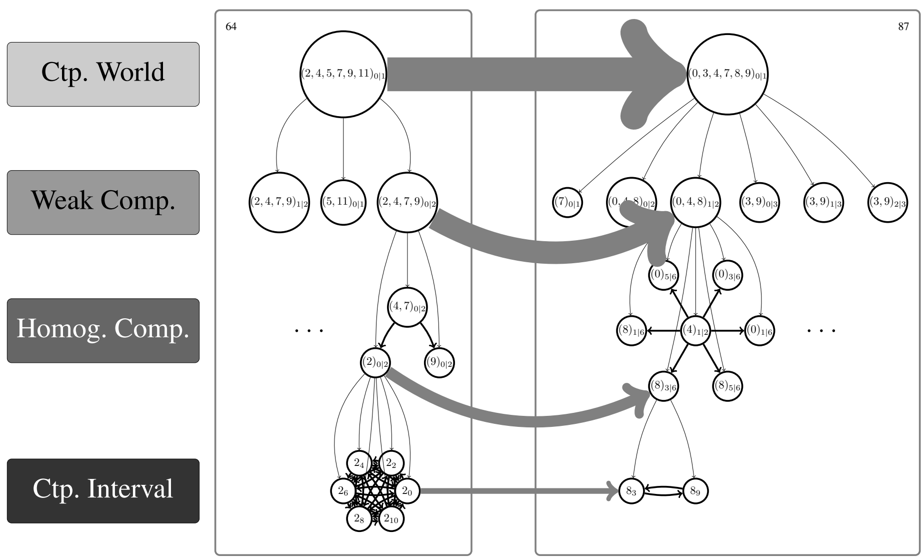


Figure 5.10: The hierarchical construction of the quotient assignment tree $T_{\mathcal{Q}a}$ between the Dur world Δ_{64} and the Fuxian world Δ_{87} (82 in *ToM*). Grey arrows represent individual quotient assignments, the root arrow appearing at the top, the lower ones being descendants. Each one is the child of the arrow laying just above. Bold black arrows show the forbidden steps at each level, and narrow arrows the hierarchical inclusions. Every quotient vertex is in fact a digraph of quotient vertices at a lower level. For readability, loops are not displayed. The designation of vertices follows the notation for quotient components introduced in (4.16).

Chapter 6

Forms and Denotators Formalism

The functorial music theory described in the *Topos of Music* proposes an architecture of concepts inspired by category theory. A *topos* can be thought of as a complex mathematical construction bringing together algebra and set theory, a combination that allows us to perform geometrical and logical operations at the same time. The universal data format used by the software presented in Chap. 7 could be developed with the help of such tools applied to computer science.

The theory of categories is an attempt to unify mathematics under a common formalism, by building up from the structural least common denominator found across different branches of mathematics, and searching for universal properties. Be it set theory, geometry or algebra, one always ends up with objects manipulated by transformations and operations, called morphisms in this context. Abstraction leads to focus on morphisms only, by replacing objects by their identity morphisms. Remember that a graph is fully described by its arrow structure, vertices being implicitly included as tails and heads.

Therefore, in functorial music theory, *static* points are left out and replaced by sets of arrows. This more *dynamic* point of view, while increasing the complexity of the formalism, has at least three advantages.

1. On the conceptual level, the Yoneda lemma can be interpreted as the fact that an object (point) is known completely once all points of view (arrows pointing at it) are known.
2. On the theoretical level, the topos of presheaves $\mathfrak{Mod}^{@}$ cumulates the advantages of set theory, whose combinatorics of subsets can serve to define new sets, and algebra, which provides a solid and familiar structure for symmetry operations.
3. On a practical level, objects we can manipulate are no more unique instances characterised by fixed values. Objects are parametrised by functions, called *addresses*, that turn them into a factory of instances. For example, a whole family of scores and notes could be generated by simply letting an address vary. After

evaluation, pitches, onsets, durations, etc. may differ, but all generated scores would share a common origin and structure.

Gérard Millmesiter implemented this formalism successfully in the Rubato Composer application, which will be used as a shell by our counterpoint manipulation software. Data exchange between different modules, or plug-ins, occurs by means of a data encapsulation scheme called *denotators*. The concepts of *counterpoint worlds* and *counterpoints* need to be formulated first in this language before instances can be exchanged with and manipulated in other processing units.

We start with the mathematical framework and see how functors (Sec. 6.2) are built on top of modules (Sec. 6.1). This category of presheaves serves for the definition of forms and their denotators, described in Sec 6.3. Sec. 6.4 eventually explains how the contrapuntal objects introduced in Chap. 2 fit into this generic architecture.

Functorial music theory is a very versatile tool, that can seamlessly be applied to algebraic counterpoint. But the full power of the theory will not be revealed in this quite limited application. Interested readers are referred to the reference text [MGM02], especially Chap. 6, or encouraged to read Milmeister's PhD thesis [Mil06], or Florian Thalmann's master's thesis [Tha07], which both offer a nice introduction to the subject.

6.1 Algebraic Category

Their strong algebraic structure and versatility makes modules and diaffine transforms the candidates of choice for serving as building blocks in functorial music theory.

6.1.1 Modules

Modules are a generalisation of vector spaces to rings. Their coverage is sufficiently broad to encompass not only any primitive type found in common programming languages, but also more sophisticated mathematical constructions such as modular integers, polynomials, word monoids, and matrices, to name only a few.

Definition 44. *An R-**module**, written M for short, is a triple $(R, M, \cdot)$ containing:*

1. *A ring of **scalars** $(R, +_R, \cdot_R)$. It is not necessarily commutative in general, but all rings encountered in this dissertation are.*
2. *An abelian group $(M, +_M)$ of **vectors**.*
3. *A **scalar multiplication** $\cdot : R \times M \to M$ having the following properties:*
   - *(a) Identity*
   $$1_R \cdot m = m \quad \forall m \in M \tag{6.1}$$
   - *(b) Left and right distributivity*
   $$(r +_R r') \cdot m = r \cdot m +_M r' \cdot m \quad \forall r, r' \in R, \forall m \in M \tag{6.2}$$
   $$r \cdot (m +_M m') = r \cdot m +_M r \cdot m' \quad \forall r \in R, \forall m, m' \in M \tag{6.3}$$

(c) *Associativity*

$$r \cdot (r' \cdot m) = (r \cdot r') \cdot m \quad \forall r, r' \in R, \forall m \in M \tag{6.4}$$

6.1.2 Diaffine Transforms

Diaffine transforms are an extension of the usual affine transforms to R-modules. Most of the current symmetry operations and transforms needed in music theory are of this type.

We construct diaffine transforms in three steps. The simplest module morphism definition handles the case where all modules are defined on a same ring.

Definition 45. *Let R be a ring. A R-**linear homomorphism** l_0 between two R modules $(R, M, \cdot_M)$ and $(R, M', \cdot_{M'})$ is a group homomorphism*

$$l_0\,(m +_M m') = l_0\,(m) +_{M'} l_0\,(m') \quad \forall m, m' \in M \tag{6.5}$$

compatible with the scalar multiplication.

$$l_0\,(r \cdot_M m) = r \cdot_{M'} l_0\,(m) \quad \forall r \in R, \forall m \in M \tag{6.6}$$

A change of rings is possible as soon as an appropriate rule of transformation is provided.

Definition 46. *Let $(R, M, \cdot_M)$ and $(R', M', \cdot_{M'})$ be two modules. A **dilinear homomorphism** $f_0 : M \to M'$ is a couple (ϕ, f) constituted by:*

1. *A ring homomorphism $\phi : R \to R'$ called **scalar restriction**.*
2. *A group homomorphism $f : (M, +_M) \to (M', +_{M'})$.*

The mapping occurs in a way compatible with the scalar restriction.

$$f_0\,(r \cdot_M m) = \phi\,(r) \cdot_{M'} f\,(m) \quad \forall r \in R, \forall m \in M \tag{6.7}$$

Linear operations describe only such transformations as scaling, rotation or inversion, which preserve the neutral element. Since it is also necessary to act on position, as happens for musical transpositions, a way of translating objects needs to be considered too.

Definition 47. *Let M and M' be any two modules. A diaffine homomorphism $f : (R, M, \cdot_M) \to (R', M', \cdot_{M'})$ is a couple $(f_0, e^{m'})$ defined by*

1. *A Dilinear homomorphism $f_0 : (R, M, \cdot_M) \to (R', M', \cdot_{M'})$.*
2. *A mapping $e^{m'}$ of the codomain called a **translation**, where $m' \in M'$.*

$$\begin{aligned} e^{m'} : M' &\longrightarrow M' \\ x &\longmapsto x +_{M'} m' \end{aligned} \tag{6.8}$$

The diaffine homomorphism results from the composition of the two elements:

$$f := e^{m'} \circ f_0. \tag{6.9}$$

Modules and diaffine transforms build the category $\mathfrak{Mod}$. The next section explains how they can be enriched further by assigning them functors.

6.2 Functorial Category

The category of modules $\mathfrak{Mod}$, while providing the algebraic structure needed for performing geometrical operations, still lacks elementary set theoretical properties useful for logic. The introduction of functors, which associate modules (*points*) with sets of diaffine transforms (*arrows*), solves the problem.

6.2.1 Presheaves

Definition 48. *A set-valued presheaf F over $\mathfrak{Mod}$ is a contravariant functor from the category of modules $\mathfrak{Mod}$ into the category of sets $\mathfrak{Set}$. For a given $M \in \mathfrak{Mod}_0$, one defines a contravariant functor $@M\,(A)$ as follows:*

1. *It associates a set of diaffine homomorphisms to every module $A \in \mathfrak{Mod}_0$.*

$$A \mapsto @M\,(A) := \left\{ f \in \mathfrak{Mod}_1 \middle| \operatorname{Dom}(f) = A \wedge \operatorname{Cod}(f) = M \right\} \tag{6.10}$$

   This set will be written $A@M$ for short.

2. *Given a diaffine homomorphism $f : A \to A' \in \mathfrak{Mod}_1$, $f@M$ defines a homomorphism transformation:*

$$\begin{aligned} f@M : A@M &\to A'@M \\ u &\mapsto u \circ f. \end{aligned} \tag{6.11}$$

$$\begin{tikzcd}
 & A \arrow[r, mapsto, "@M"] \arrow[dl, "f@M\,(u)"'] \arrow[dd, "f"'] & A@M \\
M & & \\
 & A' \arrow[r, mapsto, "@M"'] \arrow[ul, "u"] & A'@M \arrow[uu, "f@M"']
\end{tikzcd}$$

Presheaves along with their natural transforms form the topos of presheaves:

$$\mathfrak{Mod}^{@} := \mathfrak{Fun}\left(\mathfrak{Mod}^{opp}, \mathfrak{Set}\right). \tag{6.12}$$

The Category of paths in Graphs

A last word on a category that will serve in next section. Any path p in a digraph D is a succession of vertices v_i along arrows a_i,

$$v_0 \xrightarrow{a_1} v_1 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} v_{n-1} \xrightarrow{a_n} v_n \tag{6.13}$$

working exactly like morphisms: $\operatorname{Dom}(p) = v_0$ and $\operatorname{Cod}(p) = v_n$. Objects then are single vertices, i.e. lazy or identity paths leading no where else than to the starting point.

$$v_0 \tag{6.14}$$

Morphism composition is defined as the concatenation of paths.

$$v_0 \xrightarrow{a_1} v_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} v_n = v'_0 \xrightarrow{a'_1} v'_1 \xrightarrow{a'_2} \cdots \xrightarrow{a'_n} v'_{n'} \tag{6.15}$$

The associative character of this law of composition can be easily verified. Paths thus form the category $\mathfrak{Path}(D)$ for a given digraph D. A diagram $\mathcal{D}$ is a functor in $\mathfrak{Fun}\left(\mathfrak{Path}(D), \mathfrak{Mod}^{@}\right)$ which translates an abstract graphical scheme into a series of links between presheaves. This will play an important role in the definition of forms, given in Sec. 6.3.1.

6.3 The Category of Forms and Denotators

Forms and *denotators* are built on top of the topos $\mathfrak{Mod}^{@}$. They serve for describing musical concepts, respectively objects, hence the title *Topos of Music* in [MGM02]. Denotators are generalised, abstract points living in a generalised, abstract space called a form. The form merely fixes the common structure, while a denotator designates a particular coordinate in that space. To make an analogy with object oriented programming, forms work like classes and denotators play the role of their instances.

6.3.1 Forms

The formal definition of a form is given below.

Definition 49. *A **form** F is a quadruple (N_F, T_F, C_F, I_F), whose four factors are defined as follows:*

1. *The name N_F can be a denotator (see below), or more simply a character string. It indicates the meaning, or purpose, of a concept. It must be unique.*

2. *The type T_F indicates the building principle of the form. Four cases are possible.*

$$T_F \in \{\textbf{\textit{Simple}}, \textbf{\textit{Limit}}, \textbf{\textit{Colimit}}, \textbf{\textit{Power}}\} \tag{6.16}$$

   They represent a basic type, conjunction, disjunction and selection respectively.

3. *The coordinator C_F describes the structure and interplay of the form's constitutive parts, or coordinates. Its nature depends on the form's type T_F:*

   (a) ***Simple**: a module.*

   $$C_F = M \quad M \in \mathfrak{Mod}_0 \tag{6.17}$$

   (b) ***Limit**: a diagram, which, in the discrete case, yields a cartesian product. Otherwise, edges translate into constraints linking vertices.*

   $$C_F = \mathcal{D} \quad \mathcal{D} \in \mathfrak{Fun}\left(\mathfrak{Path}, \mathfrak{Mod}^{@}\right) \tag{6.18}$$

(c) ***Colimit**: a diagram, which, in the discrete case, yields a disjoint union. Otherwise, paths define equivalence relations.*

$$C_F = \mathcal{D} \quad \mathcal{D} \in \mathfrak{Fun}\left(\mathfrak{Path}, \mathfrak{Mod}^{@}\right) \tag{6.19}$$

(d) ***Power**: a form.*

$$C_F = F' \tag{6.20}$$

4. *The identifier I_F is a monomorphism of functors in $\mathfrak{Mod}^{@}$*

$$I_F : \mathcal{S} \to \mathcal{F} \tag{6.21}$$

where $\mathcal{S}$ is called space *presheave, and the* frame *presheave $\mathcal{F}$ varies along with the type T_F:*

(a) ***Simple**:*

$$\mathcal{F} = @M \quad M \in \mathfrak{Mod}_0 \tag{6.22}$$

(b) ***Limit**:*

$$\mathcal{F} = \mathrm{Lim}\,(\mathcal{D}) \tag{6.23}$$

(c) ***Colimit**:*

$$\mathcal{F} = \mathrm{Colimit}\,(\mathcal{D}) \tag{6.24}$$

(d) ***Power**:*

$$\mathcal{F} = \Omega^{Fun(C_F)} \tag{6.25}$$

where Ω is the subobject classifier.

Forms are written as follows:

$$N_F. : T_F\,(C_F)\,. \tag{6.26}$$

Note that the identifier is omitted.

Ever growing levels of complexity can be attained by constructing new concepts out of simpler ones, even using cyclic (auto-referential) definitions. Some analogies listed in Tab. 6.1 may enlighten the meaning of the four types of forms.

Table 6.1: Anlaogies for the four types of forms.

Type	Logic	Computer science
Simple	axiom	primitive type
Limit	and	`struct` (C), `record` (Pascal)
Colimit	or	`union` (C), `variable record` (Pascal)
Power		`Set` (Java)

Forms of type simple constitute the basic building blocks.

Example 26. *The parametric description of a musical note traditionally makes use of five numerical values.*

N_F	T_F	C_F	I_F
`"Onset"`	***Simple***	$@\mathbb{R}$	$i \in Nat\,(\mathcal{F}, @\mathbb{R})$
`"Pitch"`	***Simple***	$@\mathbb{Q}$	$i \in Nat\,(\mathcal{F}, @\mathbb{Q})$
`"Loudness"`	***Simple***	$@\mathbb{Z}$	$i \in Nat\,(\mathcal{F}, @\mathbb{Z})$
`"Duration"`	***Simple***	$@\mathbb{R}$	$i \in Nat\,(\mathcal{F}, @\mathbb{R})$
`"Voice"`	***Simple***	$@\mathbb{Z}$	$i \in Nat\,(\mathcal{F}, @\mathbb{Z})$

The encoding of discrete values with integers, which form a ring and not a field, enforces us to use the more general modules instead of vector spaces. Distinction between objects showing an identical mathematical structure such as onset and duration is guaranteed by different names, reflecting their different usage.

Concepts of higher complexity can be obtained by combining simpler ones.

Example 27. *For a note to be fully described, all of its parameters must be known. Its form becomes*

$$Note. : \textbf{\textit{Limit}}\,(Onset, Pitch, Loudness, Duration, Voice)\,. \tag{6.27}$$

An optional coordinator can be provided, which translates walks in a directed graph into morphisms of forms. These walks could imply equations and constraints involving different factors.

$$\begin{tikzcd}
 & Voice \arrow[dl, bend right, "a_1"'] & Onset & \\
Duration & & & Pitch \arrow[d, "a_2"] \\
 & & & Loudness \arrow[ulll, bend left, "a_0"]
\end{tikzcd} \tag{6.28}$$

An example of a different combination type is given by scores.

Example 28. *In its most basic form, a music piece is a collection of notes. Such an arbitrary set of objects of the same type is best described by the type* ***Power****.*

$$Score. : \textbf{\textit{Power}}\,(Note) \tag{6.29}$$

Example 29. *Recursive concept building is also possible in this formalism, and particularly appropriate for describing hierarchical structures such as satellite notes attached to an anchor note. The first example that comes to mind is the case of ornaments. But a similar approach holds for the counterpoint context, where notes of the discantus can be attached to corresponding notes of the cantus firmus. This idea is already contained in the definition of a contrapuntal interval, see Def. 11, and illustrated by the arrows of Fig. 1.1.*

The hierarchical organisation of notes can be pursued without restriction, anchor notes becoming satellites of other anchor notes. A macro score *formalises this process. It is defined as a set of nodes*

$$MacroScore. : \textbf{\textit{Power}}\,(Node) \tag{6.30}$$

where a node is the conjunction of an anchor and satellite notes.

$$Node. : \textbf{\textit{Limit}}\,(Note, MacroScore) \tag{6.31}$$

To avoid jumping back and forth between both definitions eternally, we need a termination condition, or fixed point. Single anchor notes without satellites will play this role, constituting nodes pointing to an empty macro score.

6.3.2 Denotators

Denotators act like coordinates of generalised points in the form space.

Definition 50. *Let $A \in \mathfrak{Mod}_0$ an address module. An A-addressed denotator D is a triple (N_D, F_D, C_D), where:*

1. *N_D is the name designating the object.*
2. *F_D is the form, i.e. the space the denotator is living in.*
3. *$C_D \in A@\mathcal{S}$ are the coordinates characterising the denotator's generalised location. It is a diaffine homomorphism from the address module A into the denotators's form space $\mathcal{S} = \mathrm{Dom}\,(I_F)$.*

The usual notation is

$$N_D : A@F_D\,(C_D)\,. \tag{6.32}$$

Example 30. *A simple denotator can be used to describe the pitch parameter of a note. A way to describe a $C\sharp$ could be done the following way:*

$$C\sharp : \mathbb{Z}@Pitch(f) \tag{6.33}$$

where the coordinates $f : \mathbb{Z} \to \mathbb{Q}$ may be any diaffine homomorphism from the integers into the rational numbers.

Many choices are possible for the coordinate function. But one kind plays a special role: the constant functions. They allow the evaluation of the parameters, an operation that corresponds to an address change from a general module M to the zero module $\{0\}$. The constant function picks up a single value in M on which the coordinate function will be evaluated, and fixes the value of the parameter, turning traditional non-functorial theory into a special case of the functorial one. Denotators with such an address are called zero-addressed. The address is then written 0, to remember initial objects in category theory [Gol84]. Following the MIDI conventions, a middle $C\sharp$ like the above example could be written $C\sharp 3 : 0@Pitch(\frac{61}{1})$.

The counterpoint theory presented in Chap. 2 relies on the algebraic properties of very specific modular integers. If the coordinate value would vary, which is possible in the general denotator framework, these properties could be destroyed. All denotators will thus be zero-addressed in the counterpoint context of this dissertation.

6.4 Counterpoint

The forms and denotators appearing in the examples of Sec. 6.3 serve for scores in general, and are those implemented in Rubato Composer. We now develop data stuctures particular to the counterpoint theory. Two kinds of information are relevant: We want to manipulate pieces of music, and also be able to verify their compliance with a particular system of rules of composition.

6.4.1 Strong Dichotomies

The most economical way to describe a global counterpoint world $\widetilde{CW}$ is to use its generating strong dichotomy Δ. Since the former univocally determines the latter, there is no need for carrying around clumsy data structures such as interdiction tables and strict digraphs. A central repository of counterpoint worlds indexed by strong dichotomies can be maintained for saving both computation time and memory storage. A strong dichotomy consists of a variable set of pitch classes, but its cardinality is directly related to their modulus.

One problem arises if one tries to define pitch classes in a straightforward way, linking a simple form directly to the module of modular integers:

$$PitchClass12. : \textbf{Simple}\,(\mathbb{Z}_{12})\,. \tag{6.34}$$

It works perfectly for the usual octave divided into 12 semitones, but we can no longer use this space with other values of the chromatic gamut size, as would be the case for quarter-tones, where $n = 24$.

We encounter the opposite of the situation pointed out in Sec. 3.1 of [Mil06]: a *unique* concept using *several* spaces. The natural solution is hence to use disjunction. Theoretically, we would need to define it on an *infinite* number of factors, since any positive and even value of n is valid. This is not possible in Rubato Composer, where the number of factors needs to be *finite*. But this restriction is not really a problem. There are computational limits imposed by the algorithms, which grow with n. And even if we could benefit from an infinite computational power, the limitation of the human ear in discriminating different pitches sets an upper, physiological, bound on the division of the octave.

We first define all special cases:

$$\begin{aligned} PitchClass_2. :&\textbf{Simple}\,(\mathbb{Z}_2)\\ PitchClass_4. :&\textbf{Simple}\,(\mathbb{Z}_4)\\ PitchClass_6. :&\textbf{Simple}\,(\mathbb{Z}_6)\\ &\vdots \end{aligned} \tag{6.35}$$

and glue them together into a single form:

$$PitchClass. : \textbf{Colimit}\,(PitchClass_2, PitchClass_4, PitchClass_6, \ldots)\,. \tag{6.36}$$

One may wonder why not use a simpler solution, like defining a pitch class as a conjunction of pitch class index and modulus:

$$\begin{aligned} PitchClassIndex.&:\textbf{Simple}\,(\mathbb{Z})\\ PitchClassModulus.&:\textbf{Simple}\,(\mathbb{Z})\\ PitchClass'.&:\textbf{Limit}\,(PitchClassIndex, PitchClassModulus) \end{aligned} \tag{6.37}$$

The reason is that the underlying mathematical structure, as well as the concept of a pitch class, are not reflected by this separation, which makes it less satisfactory on the conceptual level. Some automatic error checking on the range of arguments may also get lost. For example, how should we handle a negative modulus value? Or an odd one?

The classes of intervals bear exactly the same structure as the pitch classes, so they will differ only in the form's name. We define the specific cases

$$\begin{aligned} IntervalClass_2.&:\textbf{Simple}\,(\mathbb{Z}_2)\\ IntervalClass_4.&:\textbf{Simple}\,(\mathbb{Z}_4)\\ IntervalClass_6.&:\textbf{Simple}\,(\mathbb{Z}_6)\\ &\vdots \end{aligned} \tag{6.38}$$

and glue them together into a unique form:

$$IntervalClass.:\textbf{Colimit}\,(IntervalClass_2, IntervalClass_4, \ldots)\,. \tag{6.39}$$

A strong dichotomy is essentially a pitch class set with special properties, see Def. 9. We could define it as a set:

$$StrongDichotomy'.:\textbf{Power}\,(PitchClass)\,. \tag{6.40}$$

But this would mean ignoring some constraints, that could here too be reflected by the architecture of concepts. All pitch classes need to share the same modulus, and since we deal with a dichotomy and not an arbitrary subset, the cardinality of the factors must be equal half of the chromatic gamut size n. This means that for any given value of n, the number of factors is fixed, making a conjunction more appropriate than a set of variable length. Define each case first:

$$\begin{aligned} StrongDichotomy_2.&:\textbf{Limit}\,(PitchClass_2)\\ StrongDichotomy_4.&:\textbf{Limit}\,(PitchClass_4, PitchClass_4)\\ StrongDichotomy_6.&:\textbf{Limit}\,(PitchClass_6, PitchClass_6, PitchClass_6)\\ &\vdots \end{aligned} \tag{6.41}$$

and glue them together into a single form, the same way we did it for pitch classes.

$$\begin{aligned} StrongDichotomy.:\textbf{Colimit}(&StrongDichotomy_2,\\ &StrongDichotomy_4, StrongDichotomy_6, \ldots). \end{aligned} \tag{6.42}$$

as illustrated in Fig. 6.1.

One constraint we left out of the form's definition is what makes a subset a strong dichotomy, namely the existence of a unique polarity function. This additional requirement could be integrated into the architecture of concepts, by specifying arrows in the **Limit** diagram. Their purpose is to restrict the value factors can take. Another solution would be to define an address from $\mathbb{Z}^{\frac{n}{2}-1}$ into $\mathbb{Z}_n$ with an appropriate affine transform sending the zero and each basis vector on a consonance.

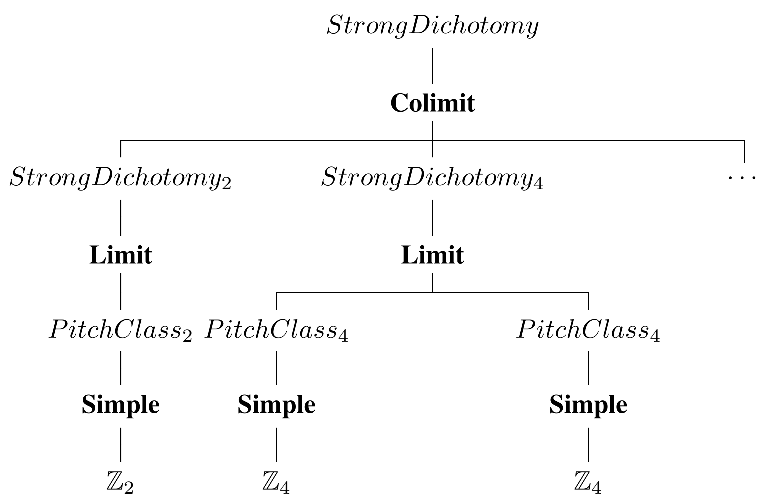


Figure 6.1: The architecture of concepts of the strong dichotomy form schematically represented as a tree.

6.4.2 Counterpoints

As explained in Def. 23 and evoked later in Sec. 4.1.1, a counterpoint is a sequence of oriented contrapuntal intervals. Four attributes suffice to describe an oriented contrapuntal interval:

1. The pitch class of the cantus firmus. Its form has already been defined in (6.36).

2. The interval class separating the discantus from the cantus firmus. Its form is similar to a pitch class, as defined in (6.39). Since cantus firmus pitch class and interval class share an indentical chromatic gamut size n, we pack them together into a single consistent form:

$$\begin{aligned} ContrapuntalInterval_{2i}. :&\textbf{Limit}(PitchClass_{2i}, \\ &IntervalClass_{2i}) \quad \forall i \in \mathbb{N}^\star \\ ContrapuntalInterval. :&\textbf{Colimit}\left(ContrapuntalInterval_2, \ldots\right). \end{aligned} \tag{6.43}$$

3. The orientation. Is it sweeping (discantus voice above cantus firmus) or hanging (discantus voice below)? The easiest way to describe the orientation is to use integers
$$Orientation. : \textbf{Simple}\,(\mathbb{Z}) \tag{6.44}$$
and observe the sign. Another possibility would be to use the finite field $\mathbb{F}_2$ in which boolean algebra can be performed, to build an indicator for hanging intervals:
$$Hanging. : \textbf{Simple}\,(\mathbb{F}_2)\,. \tag{6.45}$$

4. The step. A counterpoint is an *ordered* sequence of *variable* length. Forms of type **Limit** maintain a list of ordered factors, but are of fixed length. On the contrary, forms of type **Power** allow the number of factors to vary, but sets do not carry any notion of order. Because the sequence has to be restored in the right order, we integrate an additional attribute indicating the position of a contrapuntal interval in the counterpoint.
$$Step. : \textbf{Simple}\,(\mathbb{Z}) \tag{6.46}$$

Oriented contrapuntal intervals, defined as a conjunction

$$OrientedContrapuntalInterval. : \textbf{Limit}(ContrapuntalInterval, \\ Orientation, Step) \tag{6.47}$$

and packed into a set, form a counterpoint.

$$Counterpoint. : \textbf{Power}\,(OrientedContrapuntalInterval) \tag{6.48}$$

A graphic representation of this architecture of concepts is shown in Fig. 6.2.

The solution adopted here has an iterative flavour. Computer science teaches that every loop can also be implemented as a recursion. We could have defined an ordered list by using a circular definition: using a succession of nodes, in which a node is either (**Colimit**) an element of the list together with the list of its successors (**Limit**), or a termination element (**Simple**), e.g. the length of a list. Such a construction is at the heart of the LISP language, which treats lists exactly this way with the help of its `car` and `cdr` functions.

The $StrongDichotomy$ and $Counterpoint$ forms serve for data exchange between different modules of the software implementation presented in Chap. 7.

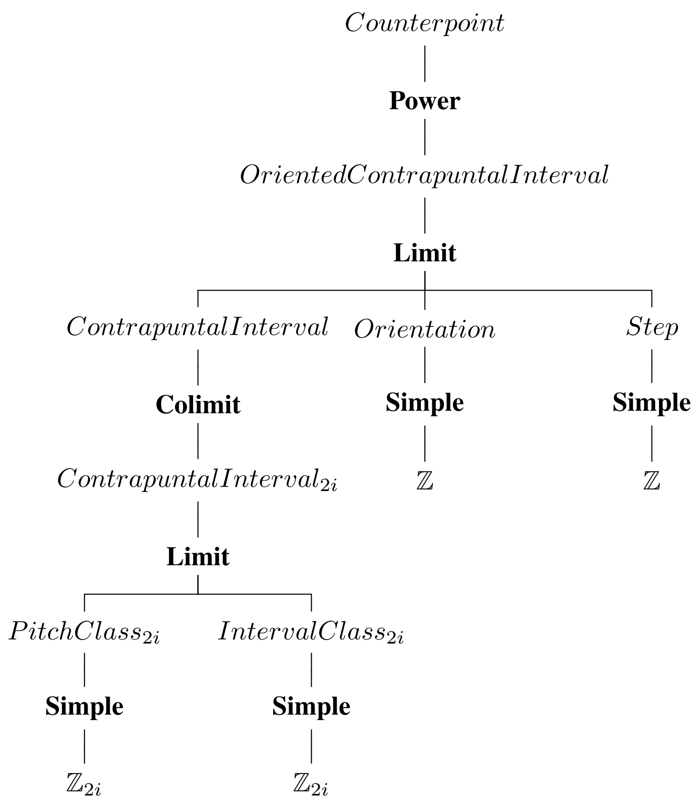


Figure 6.2: The tree representation of the counterpoint form.

Chapter 7

User Manual

Many different operations can be performed on counterpoints, such as composition, random generation, or morphing. The *BollyFux* software package provides a collection of plug-ins for the Rubato Composer platform, each one responsible for a particular task. Their combination and interplay allows the user to explore new counterpoint worlds by composing, visualising and listening to counterpoints, as illustrated in Fig. 7.1.

This chapter is intended to serve as an add-on to the Rubato Composer's reference manual [Mil06], to which the reader is referred for any questions concerning the software. Explanations about the plug-ins contained in the *BollyFux* package follow the *rubette* description scheme found in the manual.

7.1 Installation

Three pieces of software need to be installed on the user's computer:

1. A Java Runtime Environment (JRE) of version 1.5 or higher. It provides the virtual computer.
2. The Rubato Composer Java application. It provides a generic platform for combining individual software modules into a processing network, as well as an implementation of the generic theoretical framework, mathematical tools and data formats.
3. The *BollyFux* rubettes. A suite of plug-ins is responsible for creating and manipulating counterpoints.

7.1.1 Platform

If it has not already be done, Rubato Composer needs to be installed on the computer before the *BollyFux* package can be used. Rubato, along with its operating manual, can be downloaded for free.[1]

[1] Available at `http://www.Rubato.org/`.

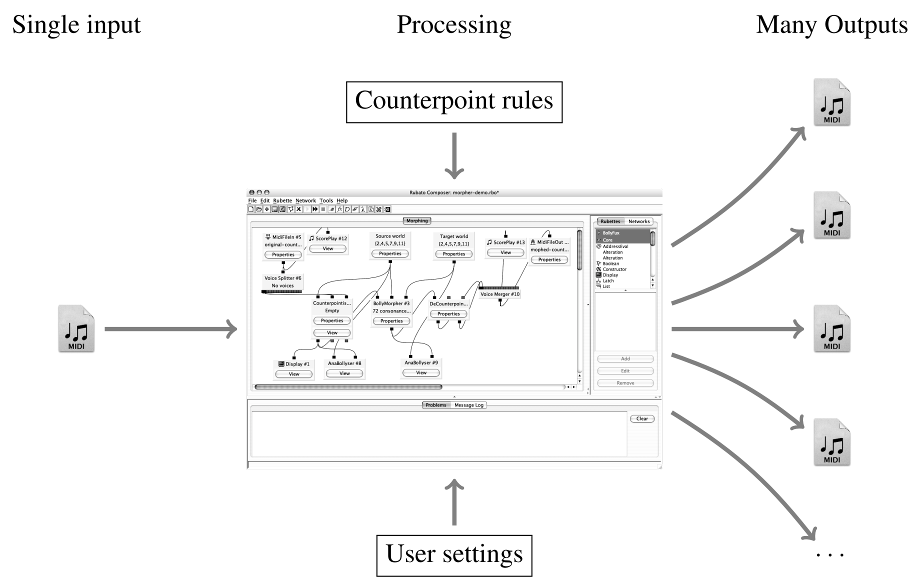


Figure 7.1: Counterpoints can be manipulated with the help of modules connected into a network in Rubato, as depicted at the center of the figure. For example, using the algorithm described in Chap. 5, a single counterpoint stored inside a MIDI file can be transformed into as many new counterpoints as there are counterpoint worlds (systems of rules) and different morphisms (assignments set by the user).

This Java application should run on any operating system equipped with a working Java Runtime Environment of at least version 1.5. It can be placed anywhere in the file system, and must be launched with help of the Java virtual machine. See your operating system's documentation for more details.

7.1.2 Plug-ins

In order to be found by Rubato and integrated at launch time, the rubettes need to reside inside a special hidden directory contained in the user's home directory. Copy the Java archive called "Bollyfux.jar" into this directory before you start the "Rubato.jar" application. On UNIX and Mac OS X systems, these operations can be achieved by issuing the following commands to the terminal:

```
# Go to the home directory
$ cd
# Create the hidden directories
$ mkdir .rubato/
$ mkdir .rubato/plugins/
# Copy the rubettes from the desktop
# (or any other location)
$ mv Desktop/BollyFux.jar .rubato/plugins/
```

See Chap. 3 of the Rubato manual for more details.

7.2 Rubettes

All *rubettes*, or plug-ins, contained in the *BollyFux* package are listed below. They enter two broad categories, those directly concerned with counterpoint are collected inside the "BollyFux Group", and the more general utilities designed for handling scores are found in the "Score Group".

Forms and *denotators* such as *StrongDichotomy* and *Counterpoint* have been specifically designed for the counterpoint manipulation. They are described in Chap. 6. Other forms such as *Score* and *Note* are of general usage and belong to the Rubato platform. Their structure is explained in the Rubato manual.

7.2.1 BollyFux Group

7.2.2 Counterpointiser Rubette

Use the *Counterpointiser* rubette to transform scores into counterpoints.

Group: BollyFux

Summary: Takes two scores, one for the cantus firmus, one for the discantus. Builds a counterpoint that can be processed by the *BollyMorpher* or *AnaBollyser* rubettes.

Inputs: One denotator of form *StrongDichotomy* and two denotators of form *Score*

1. Strong dichotomy
2. Cantus firmus
3. Discantus

Output: A denotator of form *Counterpoint*.

Description: Taking the cantus firmus and discantus scores, pairs of notes with matching onset are formed. Pitch class, interval class and orientation are used to build the succession of oriented contrapuntal intervals composing the output counterpoint. The strong dichotomy is used to set the octave division, on which the number of pitch and interval classes depend.

Properties: The properties panel is illustrated in Fig. 7.2.

- The `Reference pitch` fixes the zero pitch class $[0]_n$. Integer values correspond to MIDI pitch numbers: an indexation of semitones ranging from 0 to 127, where 60 represents the middle C, 61 the $C\sharp 3$, and so on. Micro-tonality is also possible, so that 60.5 would define a quarter-tone above C. Since counterpoint rules are invariant under global transpositions, this parameter merely affects the display of cantus firmus classes.

View: A score representation of the counterpoint.

- The `Score` panel shows the input voices in a traditional musical score representation.
- The `Table` panel contains a tabular representation of the counterpoint data: pitch and interval classes along with orientation, see Fig. 7.3. The notation for un-orientated contrapuntal intervals in which the subscript number represents the cantus firmus pitch class, and the main one the interval class, is defined in Sec. 4.2.

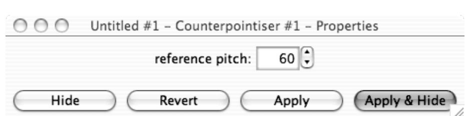


Figure 7.2: Properties panel of the *Counterpointiser* rubette.

7.2.3 DeCounterpointiser Rubette

Use the *DeCounterpointiser* rubette to transform counterpoints back into usual scores.

Group: BollyFux

Summary: Takes a counterpoint and generates two scores, the first one for the cantus firmus, the second one for the discantus.

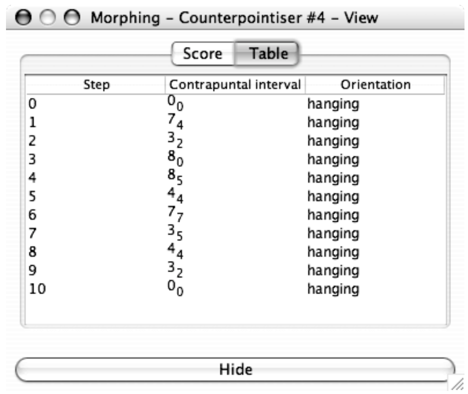


Figure 7.3: Table view of the *Counterpointiser* rubette.

Input: A denotator of form *Counterpoint*.

Outputs: Two denotators of form *Score*:

1. Cantus firmus
2. Discantus

Description: Contrapuntal intervals only store pitch and interval class information, not real pitches, that need to be chosen among all members of the class. The rubette automatically proposes an imputation for pitch values, along with other note parameters, which can be changed by hand in the properties panel.

Properties: The properties panel is illustrated in Fig. 7.4.

- The `Score` tab displays the two voices of the resulting counterpoint in the traditional musical score notation. Pitches can be changed by moving notes with the mouse.
- The `Cantus firmus` and `Discantus` tables allow direct edition of the scores by hand for fine-tuning. The five parameters of each note in the score can be altered, the values proposed ensure validity and compatibility of the data with pitch and interval classes, as well as the MIDI standard. Note that Rubato can handle micro-tonality by using fractional pitch values, while the MIDI pitch standards is limited to the usual tuning and restricts pitch to integer values.
  A first run is necessary to set the compositional rules and counterpoint. The user defined modifications of real pitch values inside pitch classes will be integrated into the output scores, but a change in the rubette's inputs will reset those values.

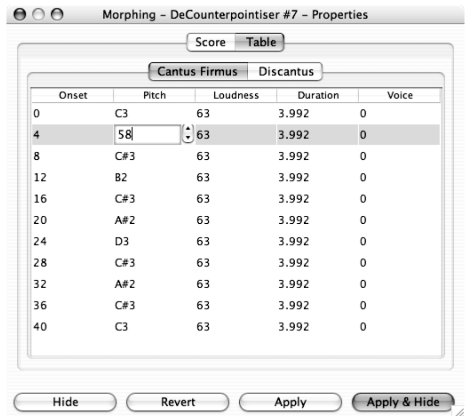


Figure 7.4: Properties panel of the *DeCounterpointiser* rubette.

7.2.4 BollyWorld Rubette

Counterpoint worlds are shaped by strong dichotomies. Use the *BollyWorld* rubette to choose the composition rules.

Group: BollyFux

Summary: Picks up a strong dichotomy and determines a contrapuntal world.

Outputs: A denotator of form *StrongDichotomy*.

Description: The lists of pitch classes that can be chosen correspond to the consonances of a strong dichotomy, that generates a counterpoint world. It fixes the rules of composition a two-voiced note against note melody has to follow, in order to become a legal counterpoint (in that particular world). It fixes the contrapuntal context in which other rubettes will work.

Properties: The properties panel is illustrated in Fig. 7.5. In the first tab, one of the six counterpoint worlds discussed in the *Topos of Music* can be chosen. In the second one, called `Custom`, any other member from their classes can be selected. The octave division can be changed to a different chromatic context, enabling the user to work with micro- or macro-tonal tunings.

- The `Octave Division` field defines the number of notes (semitones, quarter-tones, etc.) in an octave. The number of output consonances will be half the number of available notes.
- Dichotomy section
  - The `Class` popup menu selects an isometry class among the strong dichotomies. The number corresponds to the class index found in Tab. 2.2. Note that this numbering differs from the original one found in the *Topos of Music*.

  - The `Member` popup menu selects one member from the class. The list of numbers corresponds to the pitch classes of consonances.
- Constraints section
  - The `Pitch classes` check boxes force the presence of one or more pitch classes in the consonance set. Available choices in the `Member` popup reflect these constraints. The list may become empty if all constraint cannot be satisfied at the same time.
  - The `Multiplicative monoid` check box filters the consonance sets yielding a multiplicative monoid.

Higher values of the chromatic gamut size n yield a greater amount of available counterpoint worlds. They will all be generated on the fly the first time the user chooses a particular value of n. Note that the computation may take some time.

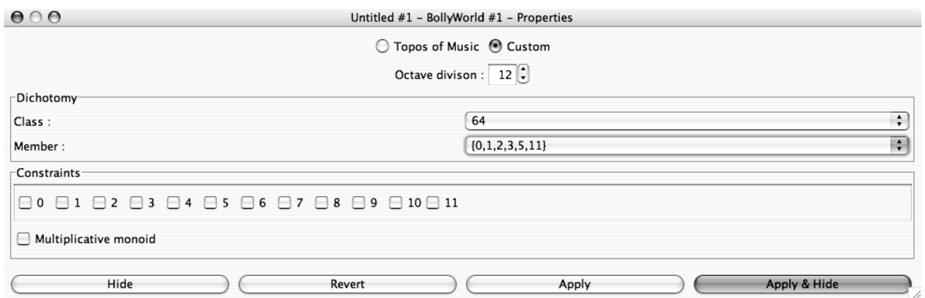


Figure 7.5: Properties panel of the *BollyWorld* rubette.

7.2.5 BollyCarlo Rubette

Traditional counterpoint exercises consist in adding a discantus to a selected cantus firmus. Let the computer do this automatically for you: The *BollyCarlo* rubette will randomly generate a discantus compliant with the rules of a given counterpoint world. This rubette is a handy tool for getting familiar with a particular counterpoint world. The ear can be trained with infinitely many counterpoint examples.

Group: BollyFux

Summary: Generates a random discantus.

Inputs: Two denotators.

1. A denotator of form *StrongDichotomy* sets the counterpoint world.
2. A denotator of form *Score* sets the cantus firmus.

Output: A denotator of form *Counterpoint*.

Description: Each note in the cantus firmus score serves as an anchor pitch class for a sweeping contrapuntal interval, whose interval is randomly generated in such a way that the resulting counterpoint is compliant with the rules effective in the given counterpoint world.

7.2.6 BollyMorpher Rubette

Use the *BollyMorpher* rubette to transform a counterpoint between two worlds. This rubette implements the algorithm discussed in Chap. 5 and allows the user to define a morphing by deciding how to map quotient graphs.

Group: BollyFux

Summary: Transforms a counterpoint from one world into another.

Inputs: Three denotators.

1. A denotator of form *Counterpoint* for the original counterpoint.
2. A denotator of form *StrongDichotomy* for the counterpoint world from which the counterpoint will be transformed (source rules).
3. A denotator of form *StrongDichotomy* for the destination world, into which the counterpoint will be transformed (target rules).

Output: A denotator of form *Counterpoint* containing the morphed counterpoint.

Description: The contrapuntal intervals of a given counterpoint are transformed in such a way that the authorisation/interdiction structure is preserved in the target world. This means that pitch and intervals classes are altered so that consonances go into consonances, dissonances into dissonances, allowed steps into allowed steps, and forbidden ones into forbidden ones.

Properties: The properties panel is illustrated in Fig. 7.6. Not all combinations of counterpoint and counterpoint worlds allow the construction of a morphism. In case of such an impossibility, the controls get disabled.

- The `Global` and `Local` radio buttons allow the user to choose between a minimal mapping, where only the contrapuntal intervals used by the counterpoint should be morphed (local), or a complete mapping in which the whole set of available contrapuntal intervals gets mapped (global).
- The list of pop-up menus let the user choose between different quotient mappings, as described in Sec. 4.3. Each pop-up shows a list of possible assignments, that may eventually lead to a valid mapping of the contrapuntal intervals. The user can construct the complete quotient morphisms step by step, from the null level down to the full level of individual contrapuntal interval mappings. All mappings impacted by a new selection will update accordingly.

- The `Reset` button clears all mappings already selected by the user. It is intended for starting over from scratch.
- The `Random` button proposes a randomly generated strict digraph morphism, i.e. a random mapping of all contrapuntal intervals.

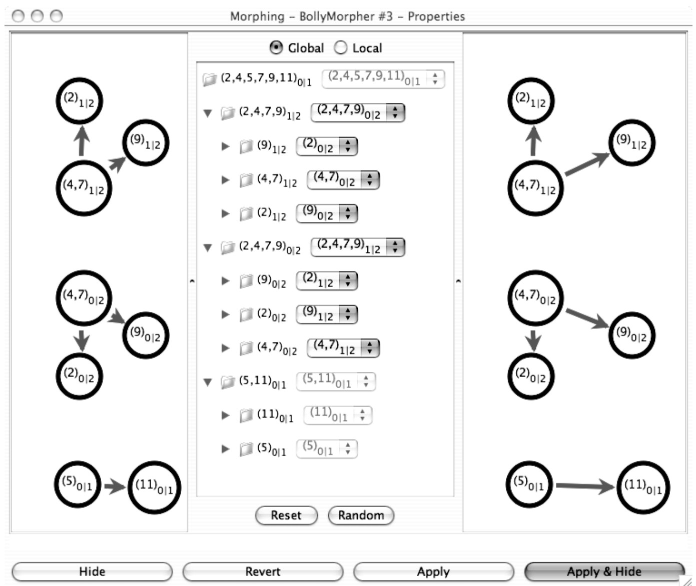


Figure 7.6: Properties panel of the *BollyMorpher* rubette. The source homogeneous quotient graph is shown on the left, the target on the right. Quotient components occupied by a counterpoint or a whole world are drawn with thick lines, as well as components in the image set. Vertices can be moved by hand to enhance readability.

How to Produce a Morphed Counterpoint

Once a network has been set up to feed the *BollyMorpher* rubette with appropriate counterpoint worlds and counterpoints, two runs, or passes, are necessary to produce an output.

1. Choose a source and a target counterpoint world in the *Bollyworld* rubettes. Load either a counterpoint from a MIDI file, or create one using the *BollyComposer* rubette.
2. A preliminary run of the network is necessary for the rubette to identify the source and target counterpoint worlds. The underlying construction of quotient graphs and hierarchical trees may take some time—and in some cases exhaust the Java runtime's memory—depending on the complexity of the worlds and counterpoint involved, see Sec. 5.5. In $\mathcal{C}_{12}$, a higher dichotomy class index often

implies a greater amount of computations, see Tab. 5.2. This step is necessary every time the input has changed. At this stage, the rubette outputs only an empty counterpoint, since the morphism has not been defined yet.

3. If the rubette has been fed with a valid input, i.e. two counterpoint worlds and a counterpoint, it knows everything it needs to guide the user in defining a counterpoint world morphism. Open the `Properties` panel, and either build a morphism yourself by selecting appropriate assignments from the pop-up menus, or press the `Random` button and generate a morphism automatically. The menu choices propose only assignments that will eventually lead to a complete valid morphism. It is not possible to leave an unfinished work: the user cannot save a morphism while there are unmapped contrapuntal intervals left. Press `Apply` once the morphism is ready.

   - Note that as long as a valid morphism has not been defined, the rubette will output an empty counterpoint.

4. Run the network again. If the input does not change, this last step uses the information computed during the first run and outputs almost instantaneously a morphed counterpoint, even if a new morphism is defined in the `Properties` panel.

7.2.7 BollyComposer Rubette

Use the *BollyComposer* rubette to create a counterpoint which follows the rules of a particular word, and is compatible with a given scale.

Group: BollyFux

Summary: Computer-assisted counterpoint composition.

Inputs: Two denotators.

1. A denotator of form *StrongDichotomy* for the counterpoint world.
2. An optional denotator of form *Score* for the scale.

Output: Three denotators.

1. A denotator of form *Counterpoint* containing the composed counterpoint.
2. A denotator of form *Score* containing the cantus firmus voice.
3. A denotator of form *Score* containing the discantus voice.

Description: The sequence of contrapuntal intervals forming a counterpoint is interactively defined by the user. Constraints are provided by the compositional rules of the input counterpoint world. An optional scale can also restrict the set of allowed pitch classes. The separate outputs for voices can serve to feed the *De-Counterpointiser* rubette and restore the exact pitch values specified on the score. A preliminary run of the rubettes network is necessary for setting the compositional constraints, namely counterpoint rules and scale.

Properties: The properties panel is shown in Fig. 7.7.

- The `Number of intervals` field specifies the length of the counterpoint.
- The score is the interactive area in which the user specifies pitch heights by moving notes with the mouse. If a scale is provided, notes will jump between degrees of the scale.
- The `Status` line informs the user whether the note being currently moved lies at a pitch allowed by the counterpoint rules or not.

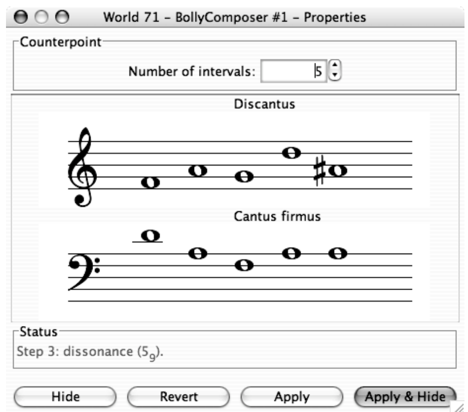


Figure 7.7: Properties panel of the *BollyComposer* rubette.

7.2.8 AnaBollyser Rubette

Use the *AnaBollyser* rubette to inspect a counterpoint for the authorised or forbidden character of each step.

Group: BollyFux

Summary: Displays the legal or illegal character of a counterpoint.

Inputs: Two denotators.

1. A denotator of form *StrongDichotomy* sets the counterpoint world.
2. A denotator of form *Counterpoint* sets the counterpoint to inspect.

Description: The contents of the counterpoint are displayed step by step, illegal elements are shown in red.

View: A table summarising the contrapuntal succession of intervals. Forbidden steps and contrapuntal dissonances are displayed in red, allowed ones in black, as shown in Fig. 7.8.

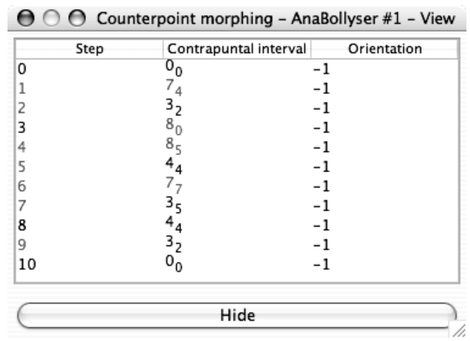


Figure 7.8: View panel of the *AnaBollyser* rubette. A red step number indicates that either the interval or the cantus firmus is forbidden. Dissonances are displayed with a red interval number. A forbidden step between two consonances is denoted by a red subscript, as well as any step involving at least one dissonance. For example, a consonance following a dissonance builds a forbidden step. Orientation is not affected by the rules.

7.2.9 Score Group

These rubettes are not directly concerned with counterpoint, but may help preparing the data for counterpoint operations. They are also part of the *BollyFux* package.

7.2.10 Voice Splitter Rubette

The *Voice Splitter* rubette can be used to separate the cantus firmus from the discantus voice in a score. This is possible as long as each one belongs to different voices of the score (the last parameter of a note). It performs the converse operation of the *Voice Merger* rubette.

Group: Score

Summary: Splits the voices of a score into separate scores, according to the note's voice parameter.

Input: A denotator of form *Score*.

Outputs: Up o sixteen denotators of form *Score*, each one containing all notes attributed to a particular voice.

Description: Notes of the input score s are sorted according to their voice parameter, and one separate score s'_j for each voice $j \in \{1, \ldots, 16\}$ is created. This operation leaves the original note parameters unchanged, it only distributes notes across different scores.

$$s : A@Score\,(c_1, \ldots, c_m) \mapsto \begin{cases} s'_1 : A@Score\left(c_{i_1^{(1)}}, \ldots, c_{i_{m_1}^{(1)}}\right) \\ \vdots \\ s'_{16} : A@Score\left(c_{i_1^{(16)}}, \ldots, c_{i_{m_{16}}^{(16)}}\right) \end{cases} \tag{7.1}$$

where the coordinates c_i are note denotators, whose five coordinates are defined in Ex. 26

$$c_i : A@Note\,(o_i, p_i, l_i, d_i, v_i)\,. \tag{7.2}$$

The indices get partitioned into 16 (possibly empty) classes

$$\uplus_{j=1}^{16}\{i_1^{(j)}, \ldots, i_{m_j}^{(j)}\} = \{1, \ldots, m\} \tag{7.3}$$

according to the voice coordinate: Note c_i is attributed to score s'_j iff the voice parameter v_i matches the output score's index j.

7.2.11 Voice Merger Rubette

The *Voice Merger* rubette can be used to create a single score containing both the cantus firmus and discantus voices. It performs the converse operation of the *Voice Splitter* rubette.

Group: Score

Summary: Merges different scores into a single one. Each one becomes a single voice of the final score.

Inputs: Sixteen denotators of form *Score*.

Output: A denotator of form *Score*.

Description: Every note is affected to the voice corresponding to its input, replacing its original voice parameter. No other note parameter is affected.

$$\left.\begin{array}{c} s_1 : A@Score\left(c_{i_1^{(1)}}, \ldots, c_{i_{m_1}^{(1)}}\right) \\ \vdots \\ s_{16} : A@Score\left(c_{i_1^{(16)}}, \ldots, c_{i_{m_{16}}^{(16)}}\right) \end{array}\right\} \mapsto s' : A@Score\,(c_1, \ldots, c_m) \tag{7.4}$$

All notes get merged into a common score. For every $i \in \{1, \ldots, m\}$ there exists a unique index combination $j \in \{1, \ldots, 16\}$ and $k \in \{i_1^{(j)}, \ldots, i_{m_j}^{(j)}\}$ such that $c_i = c_{i_k^{(j)}}$ holds.

Chapter 8

Conclusion

The classical counterpoint has been investigated and applied successfully by Western musicians during centuries. The mathematical theory of counterpoint proposes a generalisation, a broader framework in which the Fuxian counterpoint appears to be a special case.

The software tools developed in this thesis allow the exploration of new counterpoint systems, and will help to answer a fundamental question: Are the exotic counterpoint worlds as useful and interesting as the Fuxian world? If the answer turns out to be positive, then we will be able to produce new music, not by destroying old music theory and starting from scratch, but by broadening existing, well established structures.

8.1 Contribution

Achievements, benefits and by-products we obtained from our investigations lie at three levels.

Concepts

The six counterpoint worlds are no more isolated. We can connect some of them entirely, or at least find a way to transform counterpoints from one world into music compliant with the rules of another world.

Among these six counterpoint worlds in the usual twelve note-tuning, the Fuxian system occupies a special position. Although it is possible to transform some pieces of Fuxian music into counterpoints in other worlds and vice versa, it is the only system to be completely isolated from all others on a global level, i.e. there exists no global morphism transforming the Fuxian world into (or conversely from) an other counterpoint world, as shown in Fig. 5.8.

Algorithms

We added graph theory to a theoretical framework that used extensively algebra and geometry. As soon as a problem can be translated into graphs, the investigator benefits from a large amount of knowledge and algorithms.

This language is particularly efficient at expressing situations encountered when working with counterpoints: Restrictions to the intervals used by a particular counterpoint, or notes belonging to a given scale, yield induced subgraphs.

In our case, the problem of searching for a counterpoint world morphism reduces to the search for isomorphisms of subgraphs. Counterpoint morphisms can be found by using a backtracking algorithm that makes extensive usage of the particular hierarchical clustering structure of digraphs associated with counterpoint worlds.

Plane graphs also provide an excellent tool for visualising counterpoint worlds. Fig. 4.15 displays the six possible counterpoint worlds for the usual octave division $n = 12$. Such pictures give us a deeper insight into a world's particular structure, and into how algebra may shape it.

Implementation

Since these exotic worlds do not seem to have been used in any musical context so far, one question soon arises: How do they sound? In other words, how would a piece of music, composed according to these new rules, be perceived by a listener accustomed to traditional Western counterpoint?

The *BollyFux* software tools implementing the theory and concepts presented in Chap. 7 provide a direct answer to these questions. It is now possible to listen directly to counterpoints obeying exotic rules. Counterpoint worlds can be visualised and heard. Beside new ways of exploring music, computer assisted composition allows the creation of counterpoints in different worlds, or the transformation of music compliant with any system of rules.

The theory also translates seamlessly to any even division of the octave. The software makes it possible to handle micro- as well as macro-tonal counterpoints. We propose not only a theory, but also a working implementation in the Rubato Composer platform.

8.2 Summary

Some of the key theoretical achievements are enumerated below.

Strict Digraphs

Graphs of contrapuntal intervals and allowed steps, together with the usual digraph homomorphisms preserving arrows, are the graphs that may first come to mind as a model for counterpoint. Instead, we have chosen to define *strict digraphs* as those graphs whose vertices were the contrapuntal consonances and arrows the forbidden steps. *Strict morphisms* were defined more severely than the usual graph homomorphisms, since they also have to preserve the absence of arrows (mapping forbidden steps into forbidden ones).

These more restrictive versions have the theoretical advantage of allowing the construction of clean quotient structures, that greatly reduce and simplify the data to handle. On a practical level, such symmetrical structures reflect the duality between consonances and dissonances, and map the deviations of a particular counterpoint from the compositional rules faithfully.

Quotient Graphs

The most striking feature of strict digraphs is the property of their strong blocks to form cliques. The five quotient functors defined in Chap. 4 build a hierarchy and follow a progressive refinement of their respective partitions: starting with the graph as a whole and ending up with individual vertices. Each finer component is completely contained inside a single broader component. If we consider the binary relations as subsets of the contrapuntal interval set $K[\varepsilon]^2$, we can write

$$\sim_F \subseteq \sim_H \subseteq \sim_S \subseteq \sim_W \subseteq \sim_N \ . \tag{8.1}$$

A relationship that can be summarised by the following functor cascade:

$$\begin{tikzcd}
F \arrow[r, two heads, "\mathcal{H}"] \arrow[d, "\mathcal{F}\phi"'] & H \arrow[r, two heads, "\mathcal{S}"] \arrow[d, "\mathcal{H}\phi"'] & S \arrow[r, two heads, "\mathcal{W}"] \arrow[d, "\mathcal{S}\phi"'] & W \arrow[r, two heads, "\mathcal{N}"] \arrow[d, "\mathcal{W}\phi"'] & N \arrow[d, "\mathcal{N}\phi"'] \\
F' \arrow[r, two heads, "\mathcal{H}"'] & H' \arrow[r, two heads, "\mathcal{S}"'] & S' \arrow[r, two heads, "\mathcal{W}"'] & W' \arrow[r, two heads, "\mathcal{N}"'] & N'
\end{tikzcd} \tag{8.2}$$

This nice construction has been made possible by the conjunction of two facts that are direct consequences of the choice of incorporating forbidden steps instead of allowed ones into strict digraphs:

1. Reflexivity. A loop is systematically attached to each contrapuntal consonance.

2. Strong blocks form cliques. This derives from the special construction of counterpoint worlds.

This state of things brings a further, theoretical, argument in favour of the more restrictive definitions of counterpoint worlds and strict digraphs given in Chap. 3 and Sec. 4.1.

Morphism Enumeration

Enumerating all possible transformations of a counterpoint world into an other is a (usually intractable) combinatorial search problem. The algorithm we propose follows a simple backtracking approach. It works because of the special nature of strict digraphs. The orientation of arrows together with the hierarchical organisation of quotient graphs yield efficient criteria for pruning the search tree early enough, so that the task can be accomplished in a reasonable time.

Forms and Denotators

Highly versatile data formats called *forms* and *denotators* have been developed in functorial music theory [MGM02]. They easily encode the concepts developed in this dissertation. The *BollyFux* software package could be developed on the Rubato Composer platform, which uses this formalism for data exchange, storage, and retrieval.

8.3 Perspectives

Besides encouraging results, a few open questions remain and we submit some tracks on how they may be solved.

More voices

The theory discussed in Chap. 2 is a mathematical model for the very fundamental principles of counterpoint. It applies to the first step of the *Gradus ad Parnassum*, namely two-voiced note against note counterpoints. Wouldn't it be tempting to extend the theory to the following species, and handle more voices and more complex rhythmical patterns? This is not as easy as it seems. We are faced with some deep conceptual questions when trying to combine three voices or more with the help of contrapuntal intervals, which involve only two notes at a time. Should individual arrows be stacked (chaining arrows tails to heads) in order to link the voices, or on the contrary, should all arrows be attached at a common anchor note (one tail, several heads)? There is no clear answer to this question, and even musical practice and theory reflect diverging conceptions. But hierarchical structures such as the macro score introduced in Sec. 6.3 provide a clean model and could turn out to be useful. Hypergraphs in which connections involve more than two vertices may also make sense, but they have been studied far less than the usual graphs.

Higher Order Processes

As mentioned in Chap. 4, only two consecutive intervals are considered when deciding if a counterpoint step is valid. Real-world Fuxian rules involve sometimes more. One could imagine modelling processes with a longer memory, the drawback being an increasing amount of data to treat (moving from matrices to higher dimensional arrays).

Generalising the Formalism

Algebraic counterpoint theory relies on specific values and algebraic properties of modular integers and dual numbers. See for example the discussion about the role played by parameters of the contrapuntal symmetries in Sec. 2.2.2. On the other hand, graph theory is a tool whose power of abstraction is immense. It is used by the diagram functor defined in Sec. 6.3, whose purpose is to operationalise a connection scheme into any category.

If we move to graphs, the tight bound to the algebraic origins loses, and the focus shifts towards connections and relationships between objects, leaving their very precise nature in the background. In the world of functorial music theory, one could imagine meta-counterpoints defined on more abstract and general structures, not only intervals, and also integrate the possibility of general addressing offered by denotators, as pointed out in Sec. 11.3.7 of *ToM* [MGM02].

Appendix A

Strict Digraphs

The following figures show the strict digraphs (entire full graphs) of the six counterpoint worlds in $\mathcal{C}_{12}$. All of them share the same set of vertices, namely the 72 contrapuntal consonances, but their arrows vary and make each structure unique. While Fig.4.15 summarises the contrapuntal structures, Fig. A.1 to A.6 show them in complete detail.

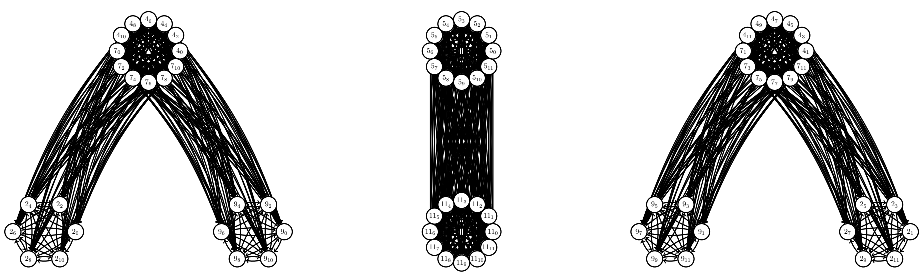


Figure A.1: The strict digraph of the Dur world Δ_{64}. For readability, loops are not shown.

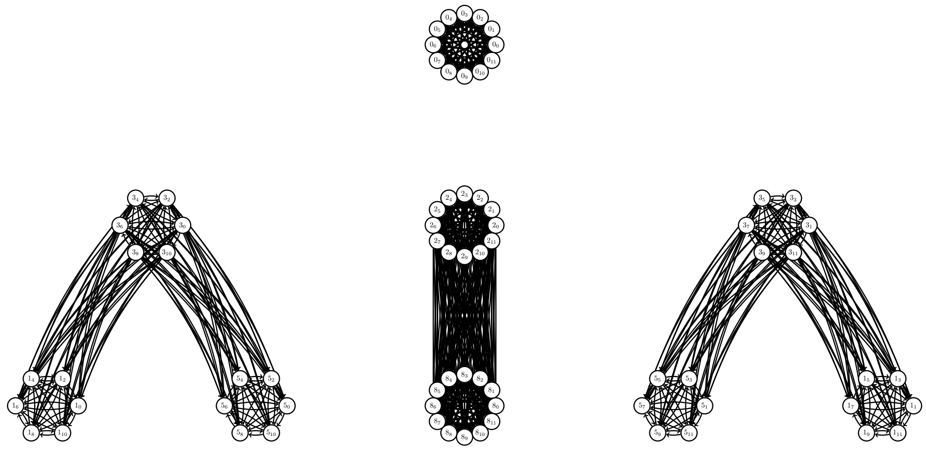


Figure A.2: The strict digraph of world Δ_{69} (68 in *ToM*). For readability, loops are not shown.

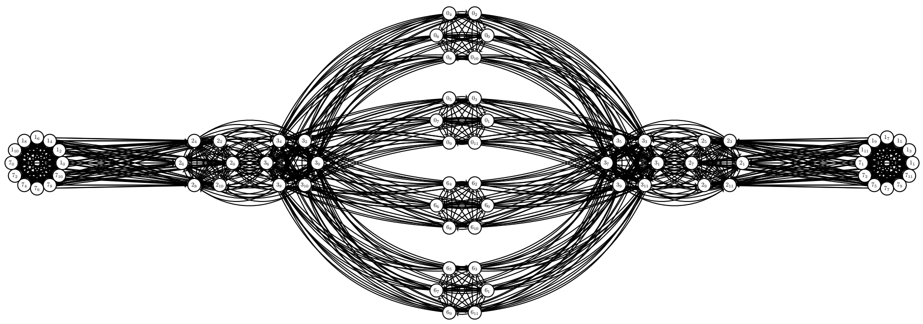


Figure A.3: The strict digraph of world Δ_{72} (71 in *ToM*). For readability, loops are not shown.

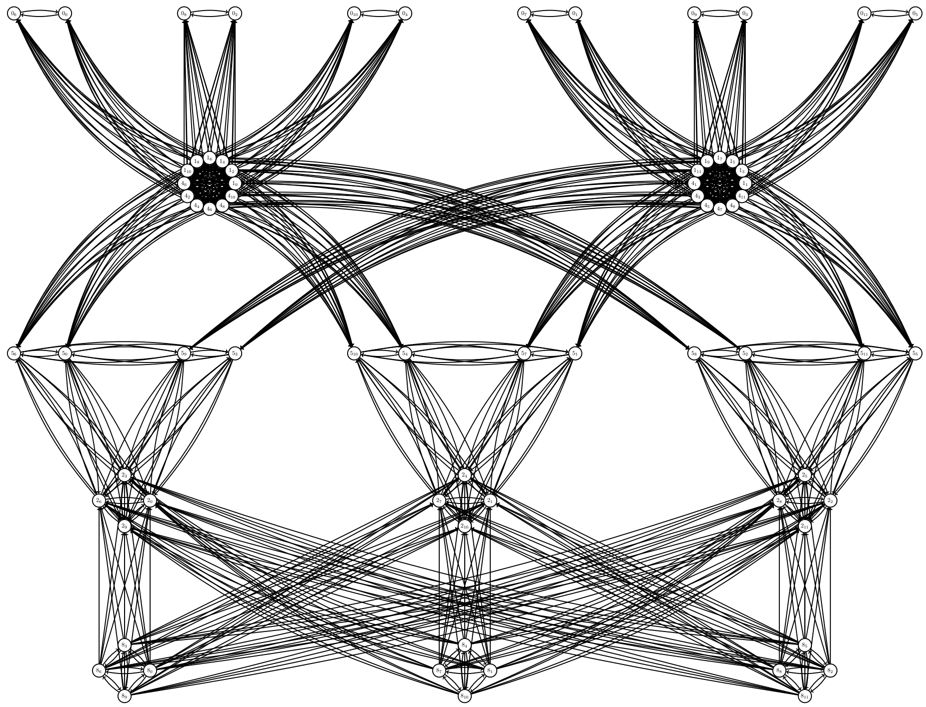


Figure A.4: The strict digraph of world Δ_{77} (75 in *ToM*). For readability, loops are not shown.

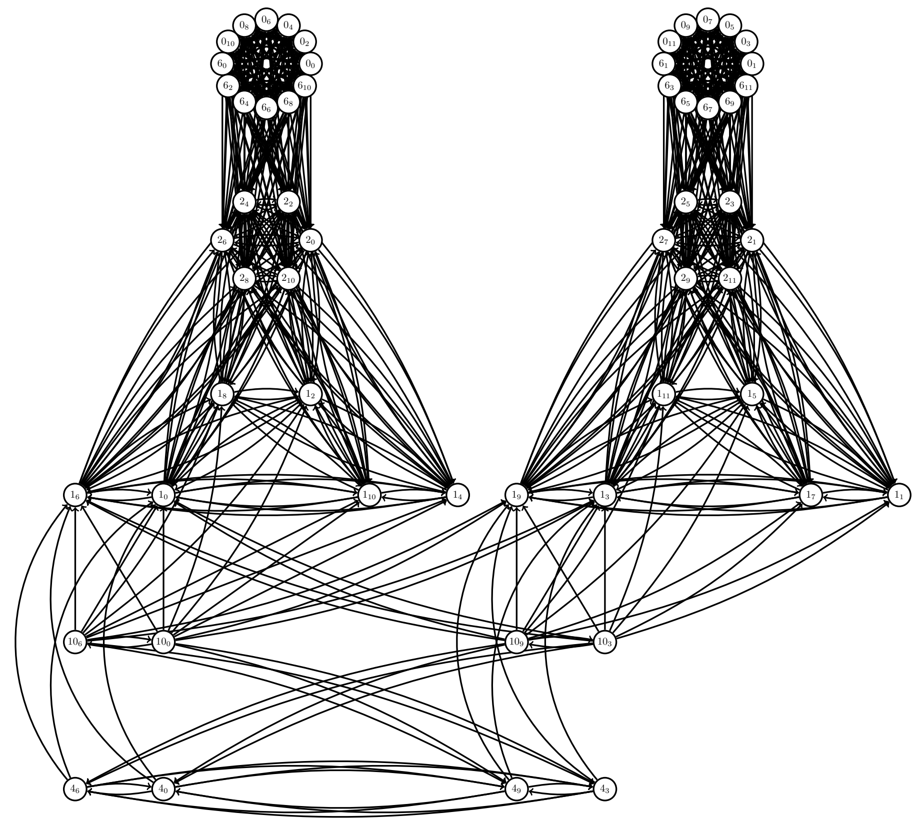


Figure A.5: The strict digraph of world Δ_{82} (77 in *ToM*). For readability, loops are not shown, as well as the structures attached to the bottom of the $(1)_{\cdot|6}$ vertices, which appear only once. The other two have been omitted and show a $(\cdot)_{1|6}/(\cdot)_{4|6}$ respectively $(\cdot)_{2|6}/(\cdot)_{5|6}$ periodicity.

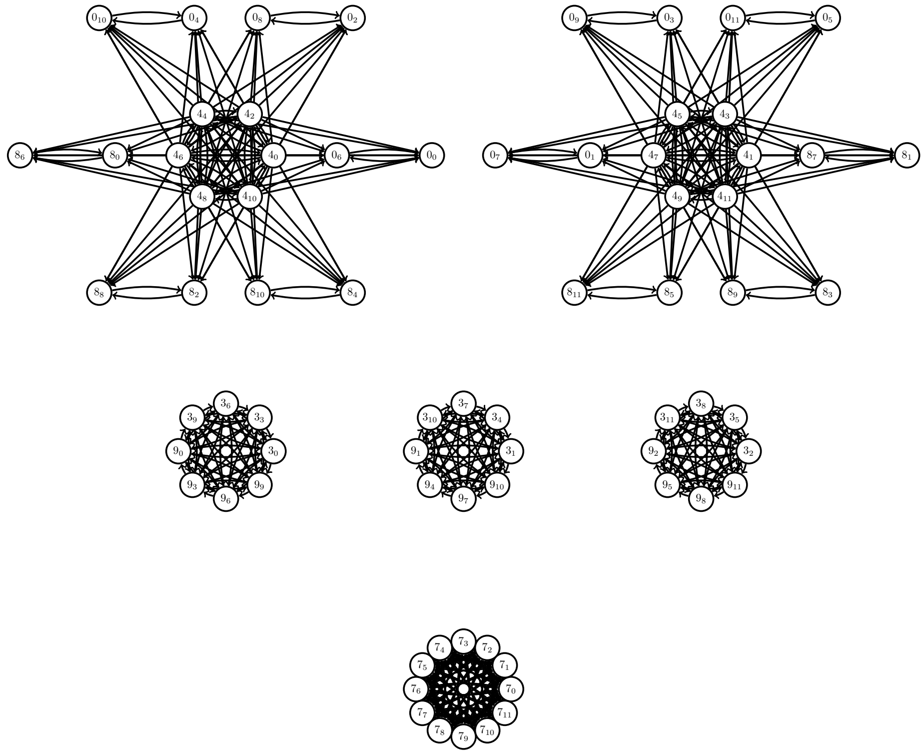


Figure A.6: The strict digraph of the Fuxian world 87 (82 in *ToM*). For readability, loops are not shown.

Appendix B

Contrapuntal Symmetries

The following tables list the contrapuntal symmetries $h \in H_{\xi_0}$ attached to a homogeneous contrapuntal consonance $\xi_0 = [0]_n + \varepsilon.k$. Tables for $n = 12$ are similar to those found in appendix O.1 of the *Topos of Music* [MGM02].

In this chapter as well as in the next ones, values of the chromatic gamut size n range from 6 to 12. We deliberately omit the lower macrotonal cases: $n = 2$, which is somehow degenerate (only one consonance exists and it cannot be repeated), and $n = 4$, for which no counterpoint worlds exist. Counterpoint worlds thus start with the whole-tone scales. We also did not include table for microtonal contexts ($n > 12$), since their number of counterpoint worlds grows quite fast. Readers interested in such exotic worlds are referred to the software presented in Chap. 7 to generate and explore them.

B.1 In $\mathcal{C}_6$

Table B.1: Homogeneous contrapuntal symmetries for strong dichotomy $(6, 7, 0)$. Consonanance k, its contrapuntal symmetries h. $\left|K[\varepsilon] \cap h\,(K[\varepsilon])\right|$ indicates the number of allowed successors for each symmetry. The total corresponds to the union of overlapping image sets. For readability, the class notation is omitted, but values have to be understood as residual classes modulo 6.

k	h	$\left\|K[\varepsilon] \cap h\,(K[\varepsilon])\right\|$
0	$e^{0+\varepsilon.2}.(5+\varepsilon.2)$	10
	$e^{0+\varepsilon.2}.(5+\varepsilon.4)$	
	Total:	14
1	$e^{0+\varepsilon.3}.(1+\varepsilon.3)$	15
	Total:	15
3	$e^{0+\varepsilon.2}.(5+\varepsilon.2)$	10
	$e^{0+\varepsilon.2}.(5+\varepsilon.4)$	
	Total:	14

B.2 In $\mathcal{C}_8$

Table B.2: Homogeneous contrapuntal symmetries for strong dichotomy $(8, 13, 0)$. Consonanance k, its contrapuntal symmetries h. $\left|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right|$ indicates the number of allowed successors for each symmetry. The total corresponds to the union of overlapping image sets. For readability, the class notation is omitted, but values have to be understood as residual classes modulo 8.

k	h	$\left\|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right\|$
0	$e^{0+\varepsilon.2}.(5+\varepsilon.2)$	20
	$e^{0+\varepsilon.2}.(5+\varepsilon.6)$	
	Total:	24
1	$e^{0+\varepsilon.4}.(1+\varepsilon.4)$	24
	Total:	24
2	$e^{0+\varepsilon.4}.(1+\varepsilon.4)$	24
	Total:	24
4	$e^{0+\varepsilon.6}.(5+\varepsilon.2)$	20
	$e^{0+\varepsilon.6}.(5+\varepsilon.6)$	
	Total:	24

Table B.3: Homogeneous contrapuntal symmetries for strong dichotomy $(8, 16, 0)$. Consonanance k, its contrapuntal symmetries h. $\left|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right|$ indicates the number of allowed successors for each symmetry. The total corresponds to the union of overlapping image sets. For readability, the class notation is omitted, but values have to be understood as residual classes modulo 8.

k	h	$\left\|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right\|$
0	$e^{0+\varepsilon.4}.(1+\varepsilon.4)$	24
	Total:	24
1	$e^{0+\varepsilon.6}.(5+\varepsilon.2)$	20
	$e^{0+\varepsilon.6}.(5+\varepsilon.6)$	
	Total:	24
3	$e^{0+\varepsilon.4}.(1+\varepsilon.4)$	24
	Total:	24
5	$e^{0+\varepsilon.2}.(5+\varepsilon.2)$	20
	$e^{0+\varepsilon.2}.(5+\varepsilon.6)$	
	Total:	24

B.3 In $\mathcal{C}_{10}$

Table B.4: Homogeneous contrapuntal symmetries for strong dichotomy $(10, 33, 0)$. Consonanance k, its contrapuntal symmetries h. $\left|K[\varepsilon] \cap h\,(K[\varepsilon])\right|$ indicates the number of allowed successors for each symmetry. The total corresponds to the union of overlapping image sets. For readability, the class notation is omitted, but values have to be understood as residual classes modulo 10.

k	h	$\left\|K[\varepsilon] \cap h\,(K[\varepsilon])\right\|$
0	$e^{0+\varepsilon.6}.(3+\varepsilon.0)$	30
	$e^{0+\varepsilon.8}.(7+\varepsilon.0)$	
	$e^{0+\varepsilon.4}.(9+\varepsilon.0)$	
	Total:	40
1	$e^{0+\varepsilon.5}.(1+\varepsilon.5)$	35
	Total:	35
2	$e^{0+\varepsilon.5}.(1+\varepsilon.5)$	35
	Total:	35
3	$e^{0+\varepsilon.5}.(1+\varepsilon.5)$	35
	Total:	35
5	$e^{0+\varepsilon.4}.(9+\varepsilon.0)$	30
	Total:	30

Table B.5: Homogeneous contrapuntal symmetries for strong dichotomy $(10, 36, 0)$. Consonanance k, its contrapuntal symmetries h. $\left|K[\varepsilon] \cap h\,(K[\varepsilon])\right|$ indicates the number of allowed successors for each symmetry. The total corresponds to the union of overlapping image sets. For readability, the class notation is omitted, but values have to be understood as residual classes modulo 10.

k	h	$\left\|K[\varepsilon] \cap h\,(K[\varepsilon])\right\|$
0	$e^{0+\varepsilon.5}.(1+\varepsilon.5)$	35
	Total:	35
1	$e^{0+\varepsilon.6}.(3+\varepsilon.2)$	34
	$e^{0+\varepsilon.6}.(3+\varepsilon.4)$	
	$e^{0+\varepsilon.6}.(3+\varepsilon.6)$	
	$e^{0+\varepsilon.6}.(3+\varepsilon.8)$	
	$e^{0+\varepsilon.8}.(7+\varepsilon.2)$	
	$e^{0+\varepsilon.8}.(7+\varepsilon.4)$	
	$e^{0+\varepsilon.8}.(7+\varepsilon.6)$	
	$e^{0+\varepsilon.8}.(7+\varepsilon.8)$	
	$e^{0+\varepsilon.4}.(9+\varepsilon.2)$	
	$e^{0+\varepsilon.4}.(9+\varepsilon.4)$	
	$e^{0+\varepsilon.4}.(9+\varepsilon.6)$	
	$e^{0+\varepsilon.4}.(9+\varepsilon.8)$	
	Total:	48
2	$e^{0+\varepsilon.5}.(1+\varepsilon.5)$	35
	Total:	35
4	$e^{0+\varepsilon.5}.(1+\varepsilon.5)$	35
	Total:	35
6	$e^{0+\varepsilon.4}.(9+\varepsilon.2)$	34
	$e^{0+\varepsilon.4}.(9+\varepsilon.4)$	
	$e^{0+\varepsilon.4}.(9+\varepsilon.6)$	
	$e^{0+\varepsilon.4}.(9+\varepsilon.8)$	
	Total:	46

Table B.6: Homogeneous contrapuntal symmetries for strong dichotomy $(10, 39, 0)$. Consonanance k, its contrapuntal symmetries h. $\left|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right|$ indicates the number of allowed successors for each symmetry. The total corresponds to the union of overlapping image sets. For readability, the class notation is omitted, but values have to be understood as residual classes modulo 10.

k	h	$\left\|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right\|$
0	$e^{0+\varepsilon.6}.(3+\varepsilon.0)$	30
	Total:	30
1	$e^{0+\varepsilon.8}.(7+\varepsilon.0)$	30
	Total:	30
2	$e^{0+\varepsilon.5}.(1+\varepsilon.5)$	45
	Total:	45
5	$e^{0+\varepsilon.6}.(3+\varepsilon.0)$	30
	Total:	30
6	$e^{0+\varepsilon.8}.(7+\varepsilon.0)$	30
	Total:	30

Table B.7: Homogeneous contrapuntal symmetries for strong dichotomy $(10, 44, 0)$. Consonanance k, its contrapuntal symmetries h. $\left|K[\varepsilon] \cap h\,(K[\varepsilon])\right|$ indicates the number of allowed successors for each symmetry. The total corresponds to the union of overlapping image sets. For readability, the class notation is omitted, but values have to be understood as residual classes modulo 10.

k	h	$\left\|K[\varepsilon] \cap h\,(K[\varepsilon])\right\|$
0	$e^{0+\varepsilon.6}.(3+\varepsilon.2)$	34
	$e^{0+\varepsilon.6}.(3+\varepsilon.4)$	
	$e^{0+\varepsilon.6}.(3+\varepsilon.6)$	
	$e^{0+\varepsilon.6}.(3+\varepsilon.8)$	
	$e^{0+\varepsilon.8}.(7+\varepsilon.2)$	
	$e^{0+\varepsilon.8}.(7+\varepsilon.4)$	
	$e^{0+\varepsilon.8}.(7+\varepsilon.6)$	
	$e^{0+\varepsilon.8}.(7+\varepsilon.8)$	
	$e^{0+\varepsilon.4}.(9+\varepsilon.2)$	
	$e^{0+\varepsilon.4}.(9+\varepsilon.4)$	
	$e^{0+\varepsilon.4}.(9+\varepsilon.6)$	
	$e^{0+\varepsilon.4}.(9+\varepsilon.8)$	
	Total:	48
1	$e^{0+\varepsilon.5}.(1+\varepsilon.5)$	35
	Total:	35
3	$e^{0+\varepsilon.5}.(1+\varepsilon.5)$	35
	Total:	35
5	$e^{0+\varepsilon.4}.(9+\varepsilon.2)$	34
	$e^{0+\varepsilon.4}.(9+\varepsilon.4)$	
	$e^{0+\varepsilon.4}.(9+\varepsilon.6)$	
	$e^{0+\varepsilon.4}.(9+\varepsilon.8)$	
	Total:	46
7	$e^{0+\varepsilon.5}.(1+\varepsilon.5)$	35
	Total:	35

Table B.8: Homogeneous contrapuntal symmetries for strong dichotomy $(10, 45, 0)$. Consonanance k, its contrapuntal symmetries h. $\left|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right|$ indicates the number of allowed successors for each symmetry. The total corresponds to the union of overlapping image sets. For readability, the class notation is omitted, but values have to be understood as residual classes modulo 10.

k	h	$\left\|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right\|$
0	$e^{0+\varepsilon.8}.(3+\varepsilon.0)$	30
	Total:	30
1	$e^{0+\varepsilon.5}.(1+\varepsilon.5)$	45
	Total:	45
3	$e^{0+\varepsilon.4}.(7+\varepsilon.0)$	30
	Total:	30
5	$e^{0+\varepsilon.8}.(3+\varepsilon.0)$	30
	Total:	30
8	$e^{0+\varepsilon.4}.(7+\varepsilon.0)$	30
	Total:	30

Table B.9: Homogeneous contrapuntal symmetries for strong dichotomy $(10, 46, 0)$. Consonanance k, its contrapuntal symmetries h. $\left|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right|$ indicates the number of allowed successors for each symmetry. The total corresponds to the union of overlapping image sets. For readability, the class notation is omitted, but values have to be understood as residual classes modulo 10.

k	h	$\left\|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right\|$
0	$e^{0+\varepsilon.5}.(1+\varepsilon.5)$	35
	Total:	35
1	$e^{0+\varepsilon.0}.(9+\varepsilon.0)$	30
	Total:	30
3	$e^{0+\varepsilon.5}.(1+\varepsilon.5)$	35
	Total:	35
6	$e^{0+\varepsilon.0}.(3+\varepsilon.0)$	30
	$e^{0+\varepsilon.0}.(7+\varepsilon.0)$	
	$e^{0+\varepsilon.0}.(9+\varepsilon.0)$	
	Total:	40
7	$e^{0+\varepsilon.5}.(1+\varepsilon.5)$	35
	Total:	35

B.4 In $\mathcal{C}_{12}$

Table B.10: Homogeneous contrapuntal symmetries for strong dichotomy Δ_{64}. Consonanance k, its contrapuntal symmetries h. $\left|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right|$ indicates the number of allowed successors for each symmetry. The total corresponds to the union of overlapping image sets. For readability, the class notation is omitted, but values have to be understood as residual classes modulo 12.

k	h	$\left\|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right\|$
2	$e^{0+\varepsilon.6}.(1+\varepsilon.6)$	48
	$e^{0+\varepsilon.8}.(5+\varepsilon.0)$	
	Total:	66
4	$e^{0+\varepsilon.6}.(1+\varepsilon.6)$	48
	Total:	48
5	$e^{0+\varepsilon.11}.(11+\varepsilon.0)$	48
	Total:	48
7	$e^{0+\varepsilon.6}.(1+\varepsilon.6)$	48
	Total:	48
9	$e^{0+\varepsilon.6}.(1+\varepsilon.6)$	48
	$e^{0+\varepsilon.3}.(7+\varepsilon.0)$	
	Total:	66
11	$e^{0+\varepsilon.8}.(5+\varepsilon.0)$	48
	$e^{0+\varepsilon.3}.(7+\varepsilon.0)$	
	$e^{0+\varepsilon.11}.(11+\varepsilon.0)$	
	Total:	60

Table B.11: Homogeneous contrapuntal symmetries for strong dichotomy Δ_{69} (68 in *ToM*). Consonanance k, its contrapuntal symmetries h. $\left|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right|$ indicates the number of allowed successors for each symmetry. The total corresponds to the union of overlapping image sets. For readability, the class notation is omitted, but values have to be understood as residual classes modulo 12.

k	h	$\left\|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right\|$
0	$e^{0+\varepsilon.6}.(7+\varepsilon.0)$	60
	Total:	60
1	$e^{0+\varepsilon.6}.(1+\varepsilon.6)$	48
	$e^{0+\varepsilon.3}.(7+\varepsilon.0)$	
	$e^{0+\varepsilon.0}.(7+\varepsilon.6)$	
	Total:	66
2	$e^{0+\varepsilon.0}.(5+\varepsilon.0)$	48
	Total:	48
3	$e^{0+\varepsilon.6}.(1+\varepsilon.6)$	48
	$e^{0+\varepsilon.0}.(7+\varepsilon.6)$	
	Total:	54
5	$e^{0+\varepsilon.6}.(1+\varepsilon.6)$	48
	$e^{0+\varepsilon.0}.(7+\varepsilon.6)$	
	$e^{0+\varepsilon.3}.(11+\varepsilon.0)$	
	Total:	66
8	$e^{0+\varepsilon.0}.(5+\varepsilon.0)$	48
	$e^{0+\varepsilon.3}.(7+\varepsilon.0)$	
	$e^{0+\varepsilon.3}.(11+\varepsilon.0)$	
	Total:	60

Table B.12: Homogeneous contrapuntal symmetries for strong dichotomy Δ_{72} (71 in *ToM*). Consonanance k, its contrapuntal symmetries h. $\left|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right|$ indicates the number of allowed successors for each symmetry. The total corresponds to the union of overlapping image sets. For readability, the class notation is omitted, but values have to be understood as residual classes modulo 12.

k	h	$\left\|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right\|$
0	$e^{0+\varepsilon.8}.(5+\varepsilon.0)$	48
	$e^{0+\varepsilon.8}.(5+\varepsilon.6)$	
	Total:	54
1	$e^{0+\varepsilon.3}.(7+\varepsilon.3)$	42
	$e^{0+\varepsilon.9}.(7+\varepsilon.3)$	
	$e^{0+\varepsilon.3}.(7+\varepsilon.9)$	
	$e^{0+\varepsilon.9}.(7+\varepsilon.9)$	
	Total:	54
2	$e^{0+\varepsilon.6}.(1+\varepsilon.6)$	60
	Total:	60
3	$e^{0+\varepsilon.6}.(1+\varepsilon.6)$	60
	Total:	60
6	$e^{0+\varepsilon.2}.(5+\varepsilon.0)$	48
	$e^{0+\varepsilon.2}.(5+\varepsilon.6)$	
	Total:	54
7	$e^{0+\varepsilon.3}.(7+\varepsilon.3)$	42
	$e^{0+\varepsilon.9}.(7+\varepsilon.3)$	
	$e^{0+\varepsilon.3}.(7+\varepsilon.9)$	
	$e^{0+\varepsilon.9}.(7+\varepsilon.9)$	
	Total:	54

Table B.13: Homogeneous contrapuntal symmetries for strong dichotomy Δ_{77} (75 in *ToM*). Consonanance k, its contrapuntal symmetries h. $\left|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right|$ indicates the number of allowed successors for each symmetry. The total corresponds to the union of overlapping image sets. For readability, the class notation is omitted, but values have to be understood as residual classes modulo 12.

k	h	$\left\|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right\|$
0	$e^{0+\varepsilon.6}.(1+\varepsilon.6)$	48
	$e^{0+\varepsilon.9}.(7+\varepsilon.0)$	
	$e^{0+\varepsilon.9}.(7+\varepsilon.4)$	
	$e^{0+\varepsilon.9}.(7+\varepsilon.8)$	
	Total:	70
1	$e^{0+\varepsilon.6}.(1+\varepsilon.6)$	48
	Total:	48
2	$e^{0+\varepsilon.8}.(5+\varepsilon.4)$	56
	$e^{0+\varepsilon.8}.(5+\varepsilon.8)$	
	Total:	64
4	$e^{0+\varepsilon.6}.(1+\varepsilon.6)$	48
	Total:	48
5	$e^{0+\varepsilon.8}.(5+\varepsilon.4)$	56
	$e^{0+\varepsilon.8}.(5+\varepsilon.8)$	
	Total:	64
8	$e^{0+\varepsilon.5}.(11+\varepsilon.0)$	48
	$e^{0+\varepsilon.5}.(11+\varepsilon.4)$	
	$e^{0+\varepsilon.5}.(11+\varepsilon.8)$	
	Total:	56

Table B.14: Homogeneous contrapuntal symmetries for strong dichotomy Δ_{82} (78 in *ToM*). Consonanance k, its contrapuntal symmetries h. $\left|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right|$ indicates the number of allowed successors for each symmetry. The total corresponds to the union of overlapping image sets. For readability, the class notation is omitted, but values have to be understood as residual classes modulo 12.

k	h	$\left\|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right\|$
0	$e^{0+\varepsilon.3}.(7+\varepsilon.3)$	42
	$e^{0+\varepsilon.9}.(7+\varepsilon.3)$	
	$e^{0+\varepsilon.3}.(7+\varepsilon.9)$	
	$e^{0+\varepsilon.9}.(7+\varepsilon.9)$	
	Total:	54
1	$e^{0+\varepsilon.6}.(1+\varepsilon.6)$	60
	Total:	60
2	$e^{0+\varepsilon.6}.(1+\varepsilon.6)$	60
	Total:	60
4	$e^{0+\varepsilon.0}.(5+\varepsilon.4)$	56
	$e^{0+\varepsilon.0}.(5+\varepsilon.8)$	
	Total:	64
6	$e^{0+\varepsilon.3}.(7+\varepsilon.3)$	42
	$e^{0+\varepsilon.9}.(7+\varepsilon.3)$	
	$e^{0+\varepsilon.3}.(7+\varepsilon.9)$	
	$e^{0+\varepsilon.9}.(7+\varepsilon.9)$	
	Total:	54
10	$e^{0+\varepsilon.6}.(5+\varepsilon.2)$	52
	$e^{0+\varepsilon.6}.(5+\varepsilon.10)$	
	Total:	60

Table B.15: Homogeneous contrapuntal symmetries for strong dichotomy Δ_{87} (82 in *ToM*). Consonanance k, its contrapuntal symmetries h. $\left|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right|$ indicates the number of allowed successors for each symmetry. The total corresponds to the union of overlapping image sets. For readability, the class notation is omitted, but values have to be understood as residual classes modulo 12.

k	h	$\left\|K[\varepsilon] \cap h\left(K[\varepsilon]\right)\right\|$
0	$e^{0+\varepsilon.6}.(1+\varepsilon.6)$	48
	$e^{0+\varepsilon.6}.(7+\varepsilon.6)$	
	$e^{0+\varepsilon.11}.(11+\varepsilon.0)$	
	$e^{0+\varepsilon.11}.(11+\varepsilon.4)$	
	$e^{0+\varepsilon.11}.(11+\varepsilon.8)$	
	Total:	70
3	$e^{0+\varepsilon.8}.(5+\varepsilon.4)$	56
	$e^{0+\varepsilon.8}.(5+\varepsilon.8)$	
	Total:	64
4	$e^{0+\varepsilon.6}.(1+\varepsilon.6)$	48
	$e^{0+\varepsilon.6}.(7+\varepsilon.6)$	
	Total:	54
7	$e^{0+\varepsilon.0}.(7+\varepsilon.0)$	60
	Total:	60
8	$e^{0+\varepsilon.6}.(1+\varepsilon.6)$	48
	$e^{0+\varepsilon.3}.(7+\varepsilon.0)$	
	$e^{0+\varepsilon.3}.(7+\varepsilon.4)$	
	$e^{0+\varepsilon.6}.(7+\varepsilon.6)$	
	$e^{0+\varepsilon.3}.(7+\varepsilon.8)$	
	Total:	70
9	$e^{0+\varepsilon.8}.(5+\varepsilon.4)$	56
	$e^{0+\varepsilon.8}.(5+\varepsilon.8)$	
	Total:	64

Appendix C

Interdiction Tables

The following tables enumerate the forbidden successors of a homogeneous contrapuntal consonance $\xi_0 = [0]_n + \varepsilon.k_0$. The cantus firmus "patterns" mentioned in Sec. 4.2 appear clearly in this almost visual representation of the step interdiction structure. The homogeneous strict digraphs shown in Fig. 4.15 carry the same message, and each representation can be deduced and constructed from the other one.

Tab. C.10 to C.15 describe the six counterpoint world in the usual $n = 12$ context. They agree with those appearing in appendix O.1 of the *Topos of Music* [MGM02], but show the complementary sets: remember that the focus is here on strict digraphs and step interdictions. The tables of the *Topos* are more detailed and list the set of *allowed* successors $h\,(K[\varepsilon]) \cap K[\varepsilon]$ for each individual homogeneous contrapuntal symmetry $h \in H_{\xi_0}$ of a contrapuntal consonance ξ_0. In order to move from the former representation to the latter, it is necessary to perform some set operations, i.e. combine the individual sets of allowed successors and take the complement of their union, or the intersection of the complements:

$$\mathcal{C}_n[\varepsilon]\backslash K_{\xi_0}[\varepsilon] = \cap_{h \in H_{\xi_0}} \left(h\,(K[\varepsilon]) \cap K[\varepsilon]\right)^c. \tag{C.1}$$

The next example illustrates this procedure.

Example 31. *Take the DUR dichotomoy Δ_{64}, whose consonances are given by the set $K = \{2, 4, 5, 7, 9, 11\}$. Let the first contrapuntal consonance, the homogeneous major second $\xi_0 = 0+\varepsilon.2$ serve as a start contrapuntal consonance. Here again, all numbers are residual classes modulo 12. The original values are taken from page 1211 of* ToM*:*

Topos of Music, Tab. O.1.1		*Quotient components*	
h	$h\,(K[\varepsilon]) \cap K[\varepsilon]$	*Allowed successors*	*Forbidden successors*
$e^6.(1+\varepsilon.6)$	*z even:* $z + \varepsilon.\{5, 11\}$	$(5, 11)_{0\|2}$	$(2, 4, 7, 9)_{0\|2} \cup (K)_{1\|2}$
	z odd: $z + \varepsilon.K$	$(K)_{1\|2}$	$(K)_{0\|2}$
$e^{\varepsilon.8}.5$	$\mathbb{Z}_{12} + \varepsilon.\{4, 5, 7, 9\}$	$(4, 5, 7, 9)_{0\|1}$	$(2, 11)_{0\|1}$

Taking the intersection of all forbidden successors, we end up with the homogeneous quotient component $(2)_{0|2}$, whose members are displayed on the first row of Tab. C.10 *by alternating crosses. As already mentioned in Sec. 2.2.3, forbidden parallels appear as rows with identical start and end consonances, and are completely filled with crosses. This happens for* $k_0 = k_1 = 5$ *and* $k_0 = k_1 = 11$ *in Tab. C.10.*

How to Read the Tables

The following tables show the forbidden steps between two successive consonances. The first column indicates the start interval, the second the end interval. The column gives the relative move of the cantus firmus.

For example, the interdiction of fifth parallels (Fig. C.1) appears in Tab. C.15 as a row completely filled with crosses. Whatever the relative steps of the cantus firmus $\forall x_1 \in \{0, \ldots, 11\}$ a fifth ($k_0 = 7$) can never move to another fifth ($k_1 = 7$).

Note that not all rows are shown. For readability, end consonances for which all cantus firmus moves are allowed, are omitted.

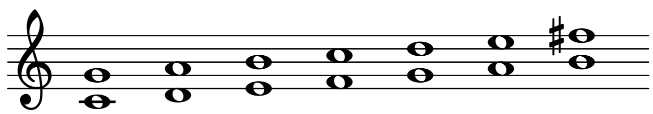


Figure C.1: Parallel movement of fifths.

C.1 In $\mathcal{C}_6$

Table C.1: Interdiction table for dichotomy $(6, 7, 0)$. Steps go from $\xi_0 = \varepsilon.k_0$ to $\xi_1 = x_1 + \varepsilon.k_1$. Numbers are residual classes modulo 6. A cross mark indicates an interdiction.

		x_1					
k_0	k_1	0	1	2	3	4	5
0	0	✖			✖		
	3	✖			✖		
1	1	✖		✖		✖	
3	0	✖			✖		
	3	✖			✖		

C.2 In $\mathcal{C}_8$

Table C.2: Interdiction table for dichotomy $(8, 13, 0)$. Steps go from $\xi_0 = \varepsilon.k_0$ to $\xi_1 = x_1 + \varepsilon.k_1$. Numbers are residual classes modulo 8. A cross mark indicates an interdiction.

		x_1							
k_0	k_1	0	1	2	3	4	5	6	7
0	0	✖				✖			
	1	✖		✖		✖		✖	
	4			✖				✖	
1	1	✖		✖		✖		✖	
	2	✖		✖		✖		✖	
2	1	✖		✖		✖		✖	
	2	✖		✖		✖		✖	
4	0			✖				✖	
	1	✖		✖		✖		✖	
	4	✖				✖			

Table C.3: Interdiction table for dichotomy $(8, 16, 0)$. Steps go from $\xi_0 = \varepsilon.k_0$ to $\xi_1 = x_1 + \varepsilon.k_1$. Numbers are residual classes modulo 8. A cross mark indicates an interdiction.

		x_1							
k_0	k_1	0	1	2	3	4	5	6	7
0	0	✖		✖		✖		✖	
	3	✖		✖		✖		✖	
1	0	✖		✖		✖		✖	
	1	✖				✖			
	5			✖				✖	
3	0	✖		✖		✖		✖	
	3	✖		✖		✖		✖	
5	0	✖		✖		✖		✖	
	1			✖				✖	
	5	✖				✖			

C.3 In $\mathcal{C}_{10}$

Table C.4: Interdiction table for dichotomy $(10, 33, 0)$. Steps go from $\xi_0 = \varepsilon.k_0$ to $\xi_1 = x_1 + \varepsilon.k_1$. Numbers are residual classes modulo 10. A cross mark indicates an interdiction.

		x_1									
k_0	k_1	0	1	2	3	4	5	6	7	8	9
0	0	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
1	1	✖		✖		✖		✖		✖	
	2	✖		✖		✖		✖		✖	
	3	✖		✖		✖		✖		✖	
2	1	✖		✖		✖		✖		✖	
	2	✖		✖		✖		✖		✖	
	3	✖		✖		✖		✖		✖	
3	1	✖		✖		✖		✖		✖	
	2	✖		✖		✖		✖		✖	
	3	✖		✖		✖		✖		✖	
5	0	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
	5	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖

Table C.5: Interdiction table for dichotomy $(10, 36, 0)$. Steps go from $\xi_0 = \varepsilon.k_0$ to $\xi_1 = x_1 + \varepsilon.k_1$. Numbers are residual classes modulo 10. A cross mark indicates an interdiction.

		x_1									
k_0	k_1	0	1	2	3	4	5	6	7	8	9
0	0	✖		✖		✖		✖		✖	
	2	✖		✖		✖		✖		✖	
	4	✖		✖		✖		✖		✖	
1	1	✖					✖				
2	0	✖		✖		✖		✖		✖	
	2	✖		✖		✖		✖		✖	
	4	✖		✖		✖		✖		✖	
4	0	✖		✖		✖		✖		✖	
	2	✖		✖		✖		✖		✖	
	4	✖		✖		✖		✖		✖	
6	1	✖					✖				
	6	✖					✖				

Table C.6: Interdiction table for dichotomy $(10, 39, 0)$. Steps go from $\xi_0 = \varepsilon.k_0$ to $\xi_1 = x_1 + \varepsilon.k_1$. Numbers are residual classes modulo 10. A cross mark indicates an interdiction.

		x_1									
k_0	k_1	0	1	2	3	4	5	6	7	8	9
0	0	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
	5	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
1	1	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
	6	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
2	2	✖		✖		✖		✖		✖	
5	0	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
	5	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
6	1	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
	6	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖

Table C.7: Interdiction table for dichotomy $(10, 44, 0)$. Steps go from $\xi_0 = \varepsilon.k_0$ to $\xi_1 = x_1 + \varepsilon.k_1$. Numbers are residual classes modulo 10. A cross mark indicates an interdiction.

		x_1									
k_0	k_1	0	1	2	3	4	5	6	7	8	9
0	0	✖					✖				
1	1	✖		✖		✖		✖		✖	
	3	✖		✖		✖		✖		✖	
	7	✖		✖		✖		✖		✖	
3	1	✖		✖		✖		✖		✖	
	3	✖		✖		✖		✖		✖	
	7	✖		✖		✖		✖		✖	
5	0	✖					✖				
	5	✖					✖				
7	1	✖		✖		✖		✖		✖	
	3	✖		✖		✖		✖		✖	
	7	✖		✖		✖		✖		✖	

Table C.8: Interdiction table for dichotomy $(10, 45, 0)$. Steps go from $\xi_0 = \varepsilon.k_0$ to $\xi_1 = x_1 + \varepsilon.k_1$. Numbers are residual classes modulo 10. A cross mark indicates an interdiction.

		x_1									
k_0	k_1	0	1	2	3	4	5	6	7	8	9
0	0	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
	5	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
1	1	✖		✖		✖		✖		✖	
3	3	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
	8	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
5	0	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
	5	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
8	3	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
	8	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖

Table C.9: Interdiction table for dichotomy $(10, 46, 0)$. Steps go from $\xi_0 = \varepsilon.k_0$ to $\xi_1 = x_1 + \varepsilon.k_1$. Numbers are residual classes modulo 10. A cross mark indicates an interdiction.

		x_1									
k_0	k_1	0	1	2	3	4	5	6	7	8	9
0	0	✖		✖		✖		✖		✖	
	3	✖		✖		✖		✖		✖	
	7	✖		✖		✖		✖		✖	
1	1	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
	6	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
3	0	✖		✖		✖		✖		✖	
	3	✖		✖		✖		✖		✖	
	7	✖		✖		✖		✖		✖	
6	6	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
7	0	✖		✖		✖		✖		✖	
	3	✖		✖		✖		✖		✖	
	7	✖		✖		✖		✖		✖	

C.4 In $\mathcal{C}_{12}$

Table C.10: Interdiction table for dichotomy Δ_{64}. Steps go from $\xi_0 = \varepsilon.k_0$ to $\xi_1 = x_1 + \varepsilon.k_1$. Numbers are residual classes modulo 12. A cross mark indicates an interdiction.

		x_1											
k_0	k_1	0	1	2	3	4	5	6	7	8	9	10	11
2	2	✖		✖		✖		✖		✖		✖	
4	2	✖		✖		✖		✖		✖		✖	
	4	✖		✖		✖		✖		✖		✖	
	7	✖		✖		✖		✖		✖		✖	
	9	✖		✖		✖		✖		✖		✖	
5	5	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
	11	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
7	2	✖		✖		✖		✖		✖		✖	
	4	✖		✖		✖		✖		✖		✖	
	7	✖		✖		✖		✖		✖		✖	
	9	✖		✖		✖		✖		✖		✖	
9	9	✖		✖		✖		✖		✖		✖	
11	11	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖

Table C.11: Interdiction table for dichotomy Δ_{69} (68 in *ToM*). Steps go from $\xi_0 = \varepsilon.k_0$ to $\xi_1 = x_1 + \varepsilon.k_1$. Numbers are residual classes modulo 12. A cross mark indicates an interdiction.

		x_1											
k_0	k_1	0	1	2	3	4	5	6	7	8	9	10	11
0	0	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
1	1	✖		✖		✖		✖		✖		✖	
2	2	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
	8	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
3	1	✖		✖		✖		✖		✖		✖	
	3	✖		✖		✖		✖		✖		✖	
	5	✖		✖		✖		✖		✖		✖	
5	5	✖		✖		✖		✖		✖		✖	
8	8	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖

Table C.12: Interdiction table for dichotomy Δ_{72} (71 in *ToM*). Steps go from $\xi_0 = \varepsilon.k_0$ to $\xi_1 = x_1 + \varepsilon.k_1$. Numbers are residual classes modulo 12. A cross mark indicates an interdiction.

		x_1											
k_0	k_1	0	1	2	3	4	5	6	7	8	9	10	11
0	0	✖		✖		✖		✖		✖		✖	
	3	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
1	1	✖		✖		✖		✖		✖		✖	
	2	✖		✖		✖		✖		✖		✖	
	7	✖		✖		✖		✖		✖		✖	
2	2	✖		✖		✖		✖		✖		✖	
	3	✖		✖		✖		✖		✖		✖	
3	2	✖		✖		✖		✖		✖		✖	
	3	✖		✖		✖		✖		✖		✖	
6	3	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
	6	✖		✖		✖		✖		✖		✖	
7	1	✖		✖		✖		✖		✖		✖	
	2	✖		✖		✖		✖		✖		✖	
	7	✖		✖		✖		✖		✖		✖	

Table C.13: Interdiction table for dichotomy class Δ_{77} (75 in *ToM*). Steps go from $\xi_0 = \varepsilon.k_0$ to $\xi_1 = x_1 + \varepsilon.k_1$. Numbers are residual classes modulo 12. A cross mark indicates an interdiction.

								x_1					
k_0	k_1	0	1	2	3	4	5	6	7	8	9	10	11
0	0	✖						✖					
1	0	✖		✖		✖		✖		✖		✖	
	1	✖		✖		✖		✖		✖		✖	
	4	✖		✖		✖		✖		✖		✖	
	5	✖		✖		✖		✖		✖		✖	
2	2	✖			✖			✖			✖		
	5	✖			✖			✖			✖		
4	0	✖		✖		✖		✖		✖		✖	
	1	✖		✖		✖		✖		✖		✖	
	4	✖		✖		✖		✖		✖		✖	
	5	✖		✖		✖		✖		✖		✖	
5	2	✖			✖			✖			✖		
	5	✖			✖			✖			✖		
8	2	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
	8	✖			✖			✖			✖		

Table C.14: Interdiction table for dichotomy Δ_{82} (78 in *ToM*). Steps go from $\xi_0 = \varepsilon.k_0$ to $\xi_1 = x_1 + \varepsilon.k_1$. Numbers are residual classes modulo 12. A cross mark indicates an interdiction.

		x_1											
k_0	k_1	0	1	2	3	4	5	6	7	8	9	10	11
0	0	✖		✖		✖		✖		✖		✖	
	2	✖		✖		✖		✖		✖		✖	
	6	✖		✖		✖		✖		✖		✖	
1	1	✖		✖		✖		✖		✖		✖	
	2	✖		✖		✖		✖		✖		✖	
2	1	✖		✖		✖		✖		✖		✖	
	2	✖		✖		✖		✖		✖		✖	
4	1	✖			✖			✖			✖		
	4	✖			✖			✖			✖		
6	0	✖		✖		✖		✖		✖		✖	
	2	✖		✖		✖		✖		✖		✖	
	6	✖		✖		✖		✖		✖		✖	
10	1	✖		✖	✖	✖		✖		✖	✖	✖	
	4				✖						✖		
	10	✖						✖					

Table C.15: Interdiction table for dichotomy class Δ_{87} (82 in *ToM*). Steps go from $\xi_0 = \varepsilon.k_0$ to $\xi_1 = x_1 + \varepsilon.k_1$. Numbers are residual classes modulo 12. A cross mark indicates an interdiction.

								x_1					
k_0	k_1	0	1	2	3	4	5	6	7	8	9	10	11
0	0	✖						✖					
3	3	✖			✖			✖			✖		
	9	✖			✖			✖			✖		
4	0	✖		✖		✖		✖		✖		✖	
	4	✖		✖		✖		✖		✖		✖	
	8	✖		✖		✖		✖		✖		✖	
7	7	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖	✖
8	8	✖						✖					
9	3	✖			✖			✖			✖		
	9	✖			✖			✖			✖		

Appendix D

Scales and Counterpoint Worlds

D.1 Compatibility Degrees

The scale numbering in Tab. D.1 to D.7 indicates the scale's dihedral class index.[1] The diatonic (or pentatonic) scale has number 1. Index 2 corresponds to the minor melodic, 4 to the unitonic and 5 to the double harmonic. Their pentatonic complements have corresponding indices. Higher numbers are less common in the Western tradition, but may be found in the Indian raga scales. At the other end, number 38 is the chromatic cluster. Tab. D.1 shows the correspondence between this numbering and the affine chord classes listed in appendix L.1 on pages 1171–2 in [MGM02].

When scales are not symmetric, inverse pairs can be distinguished by a $+$ or $-$ sign. For example, 3_- is the minor harmonic and 3_+ the major harmonic. But such a distinction is not relevant here: the compatibility degrees ν and α are invariant under an orientation change of the contrapuntal interval, or an inversion of the scale. This is due to the fact that we consider the cardinality of entire sets of pitch classes and not their particular value. For example, the total amount of discantus compatible with a given scale $\mathcal{S}$ does not change with the orientation:

$$\big|\mathcal{S} \cap \{\mathcal{S}+k\}\big| = \big|\{\mathcal{S}-k\} \cap \mathcal{S}\big|. \tag{D.1}$$

Of course, the individual pitch classes are not necessarily the same in general, as can be seen in Tab. D.2 to D.9:

$$\Big\{x \in \mathcal{S} \Big| x+k \in \mathcal{S}\Big\} \neq \Big\{x \in \mathcal{S} \Big| x-k \in \mathcal{S}\Big\}. \tag{D.2}$$

But such differences are do not appear in the rather coarse measures ν and α, and we display only the dihedral class of a scale.

[1] See `http://www.diatonic.ch/` for an explanation about this nomenclature.

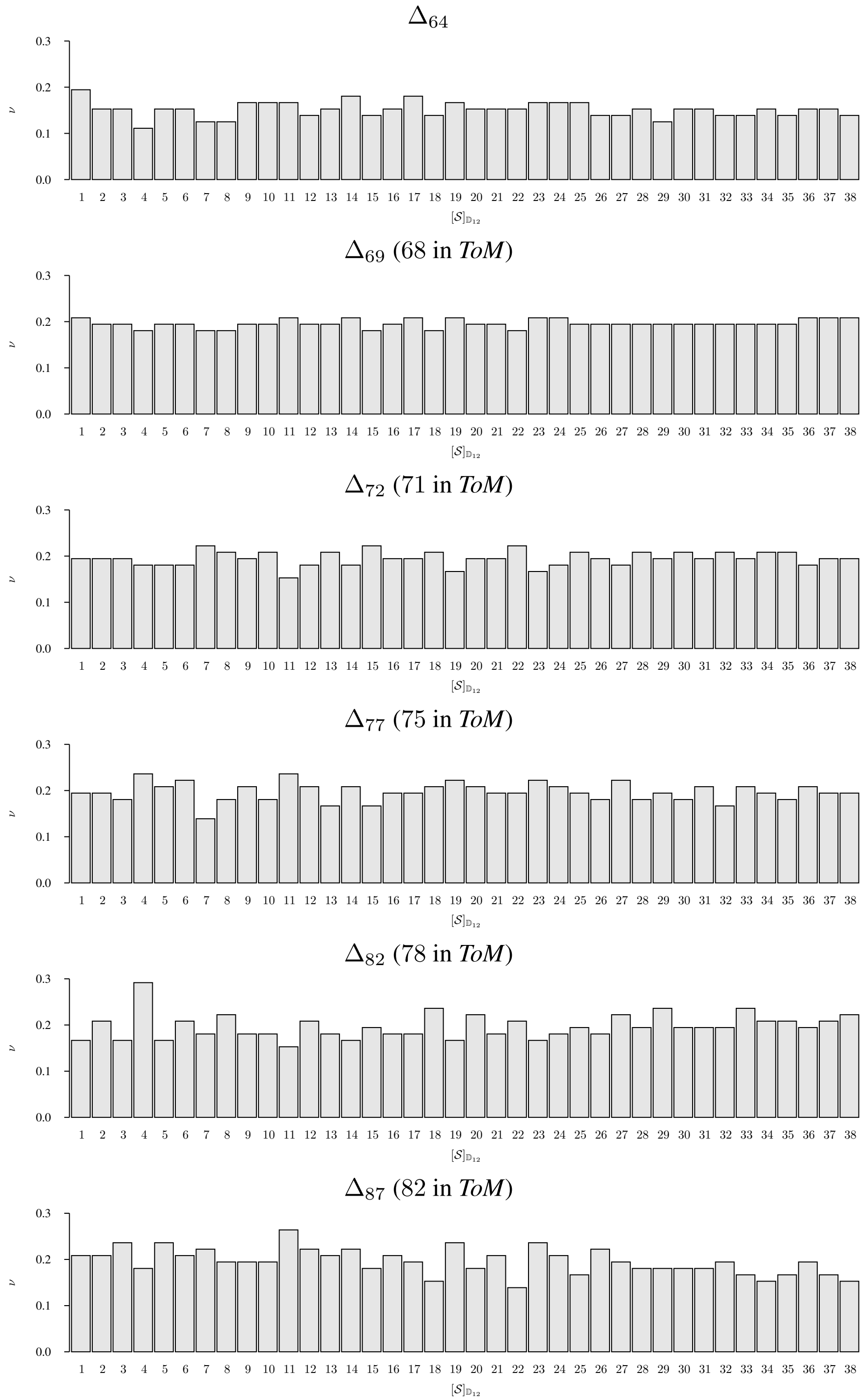


Figure D.1: The consonance compatibility degree ν of the six counterpoint worlds in $\mathcal{C}_{12}$ versus the 38 dihedral classes of pentatonic scales.

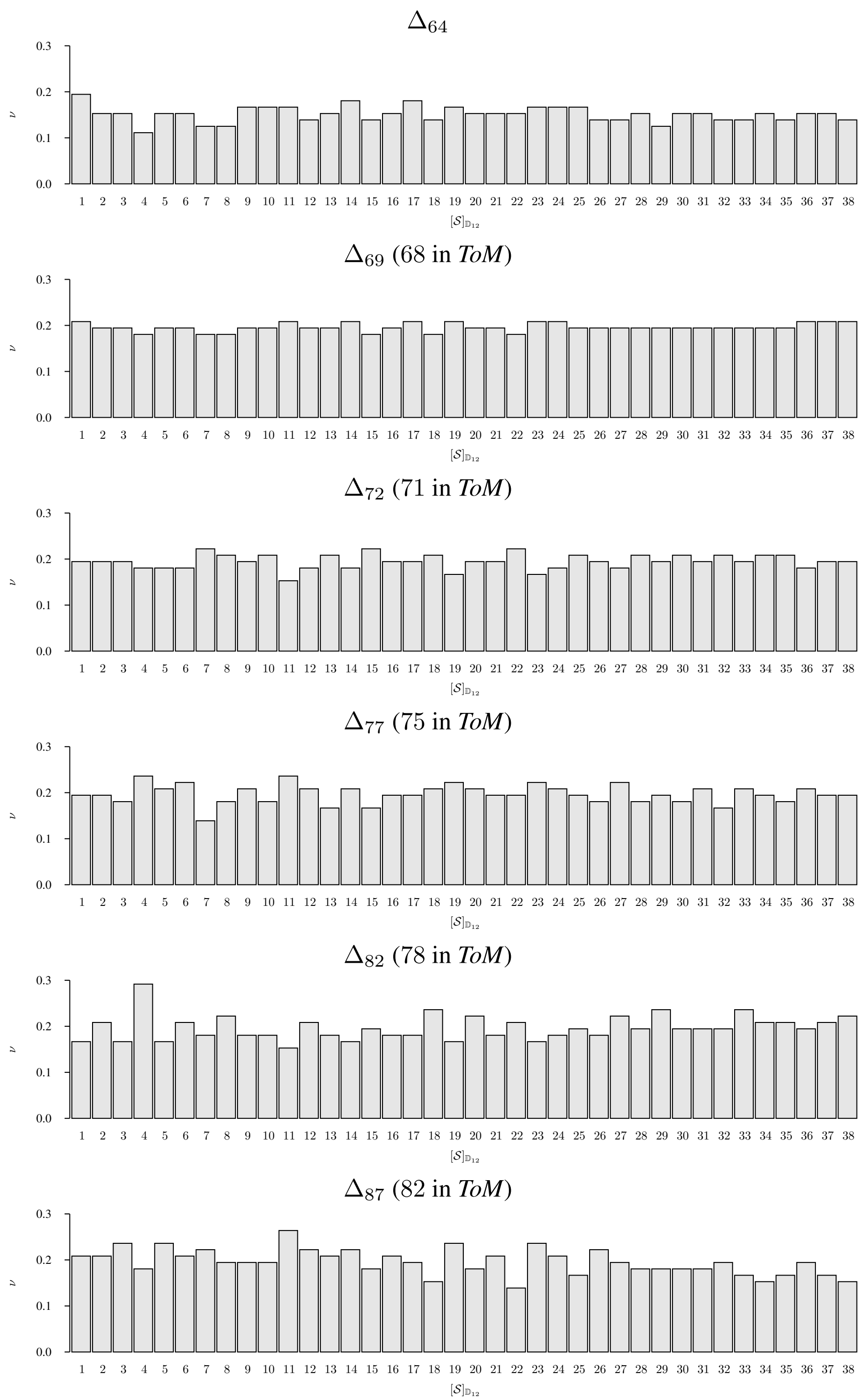


Figure D.2: The consonance compatibility degree ν of the six counterpoint worlds in $\mathcal{C}_{12}$ versus the 38 dihedral classes of pentatonic scales, for *sweeping* orientations only. The same values hold for *hanging* orientations only.

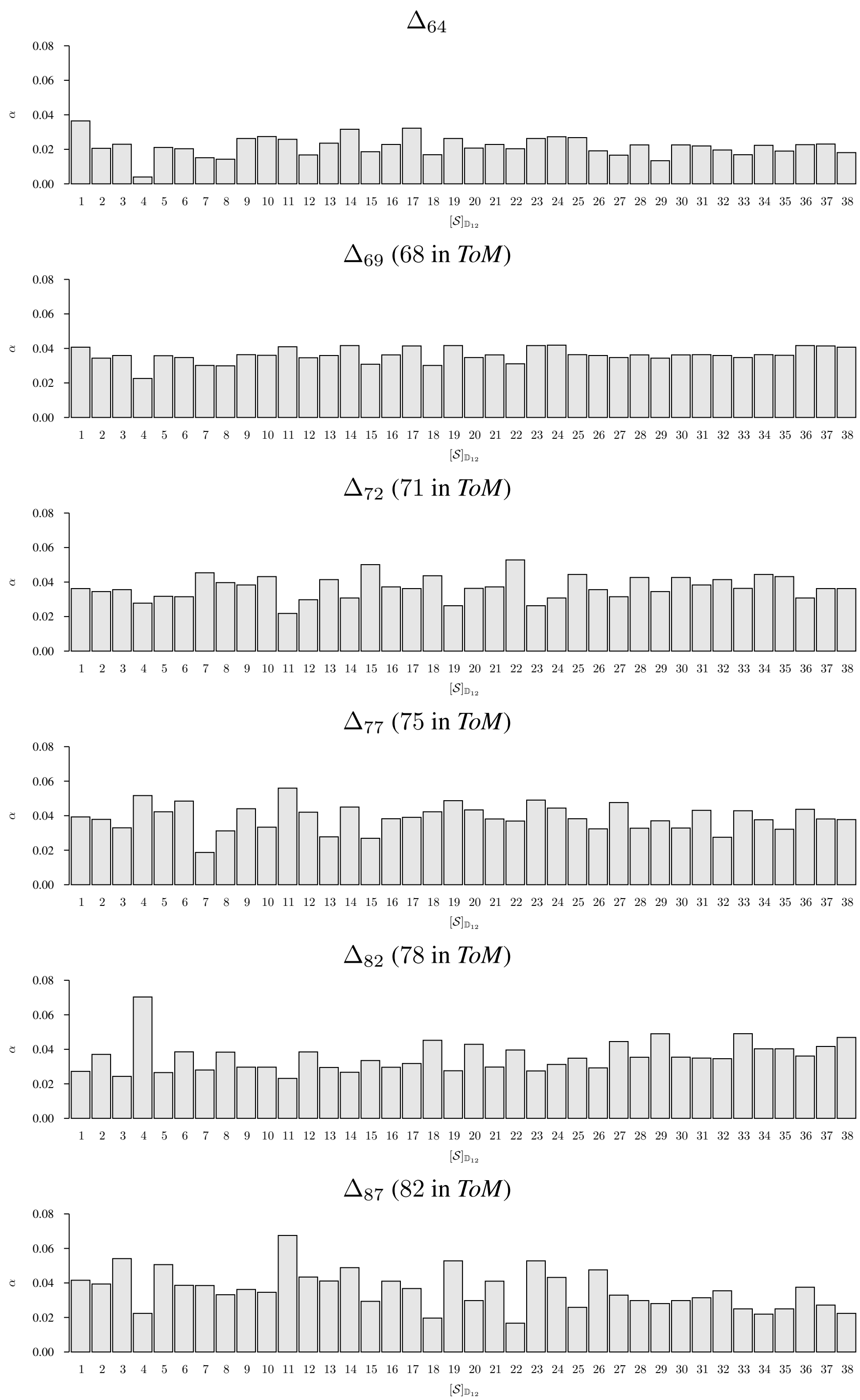


Figure D.3: The allowed step compatibility degree α of the six counterpoint worlds in $\mathcal{C}_{12}$ versus the 38 dihedral classes of pentatonic scales.

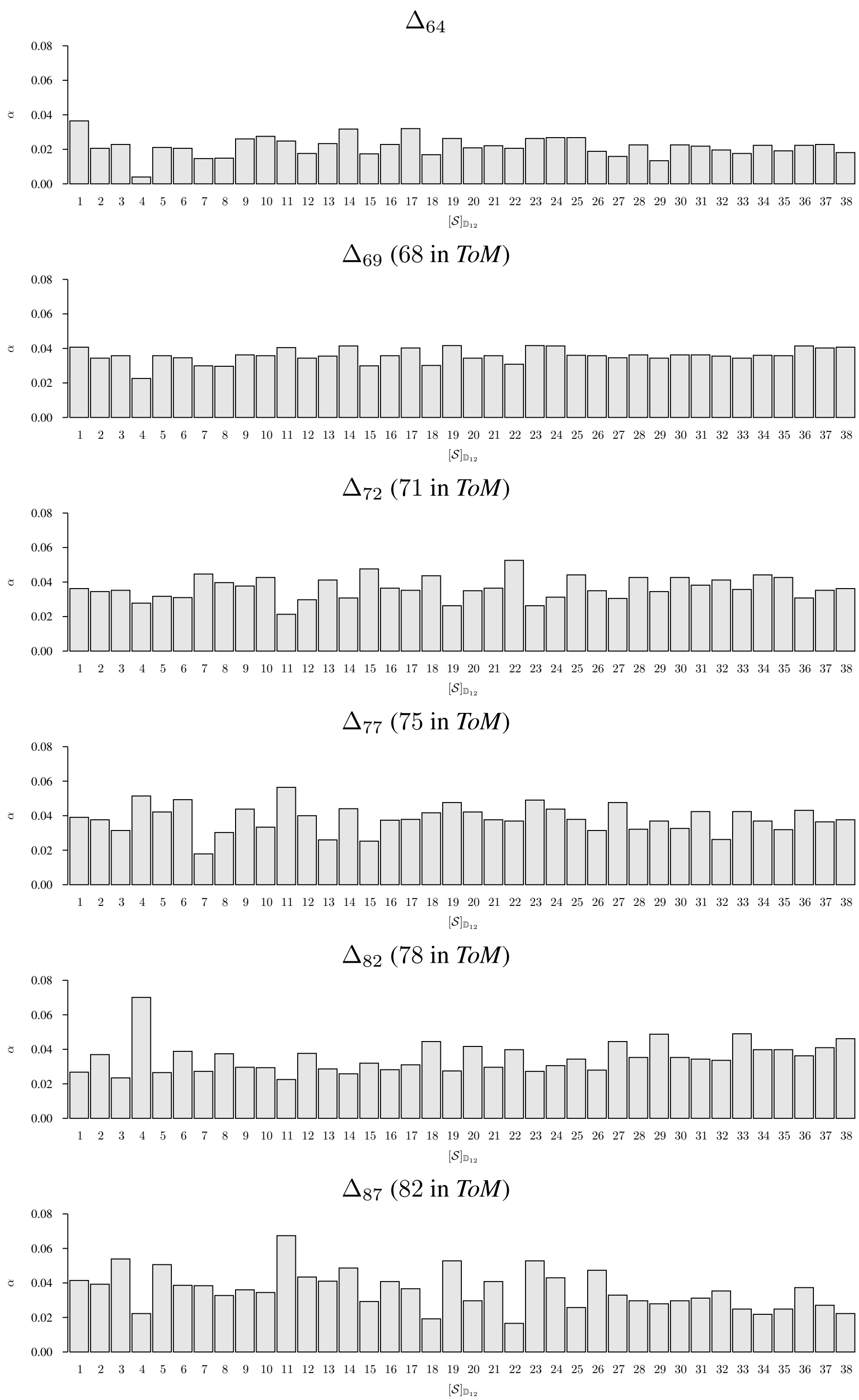


Figure D.4: The allowed step compatibility degree α of the six counterpoint worlds in $\mathcal{C}_{12}$ versus the 38 dihedral classes of pentatonic scales, for *sweeping* orientations only. The same values hold for *hanging* orientations only.

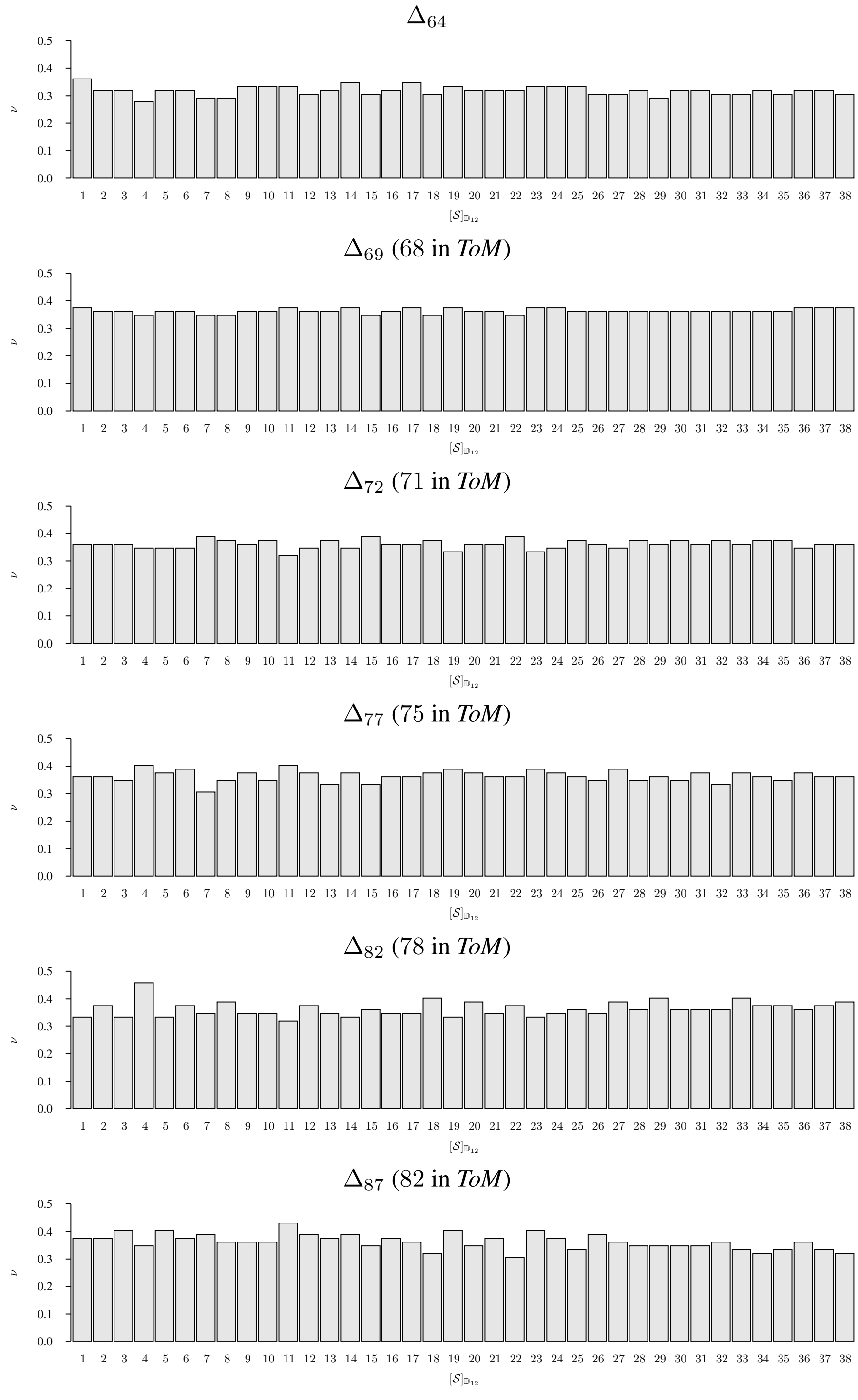


Figure D.5: The consonance compatibility degree ν of the six counterpoint worlds in $\mathcal{C}_{12}$ versus the 38 dihedral classes of heptatonic scales.

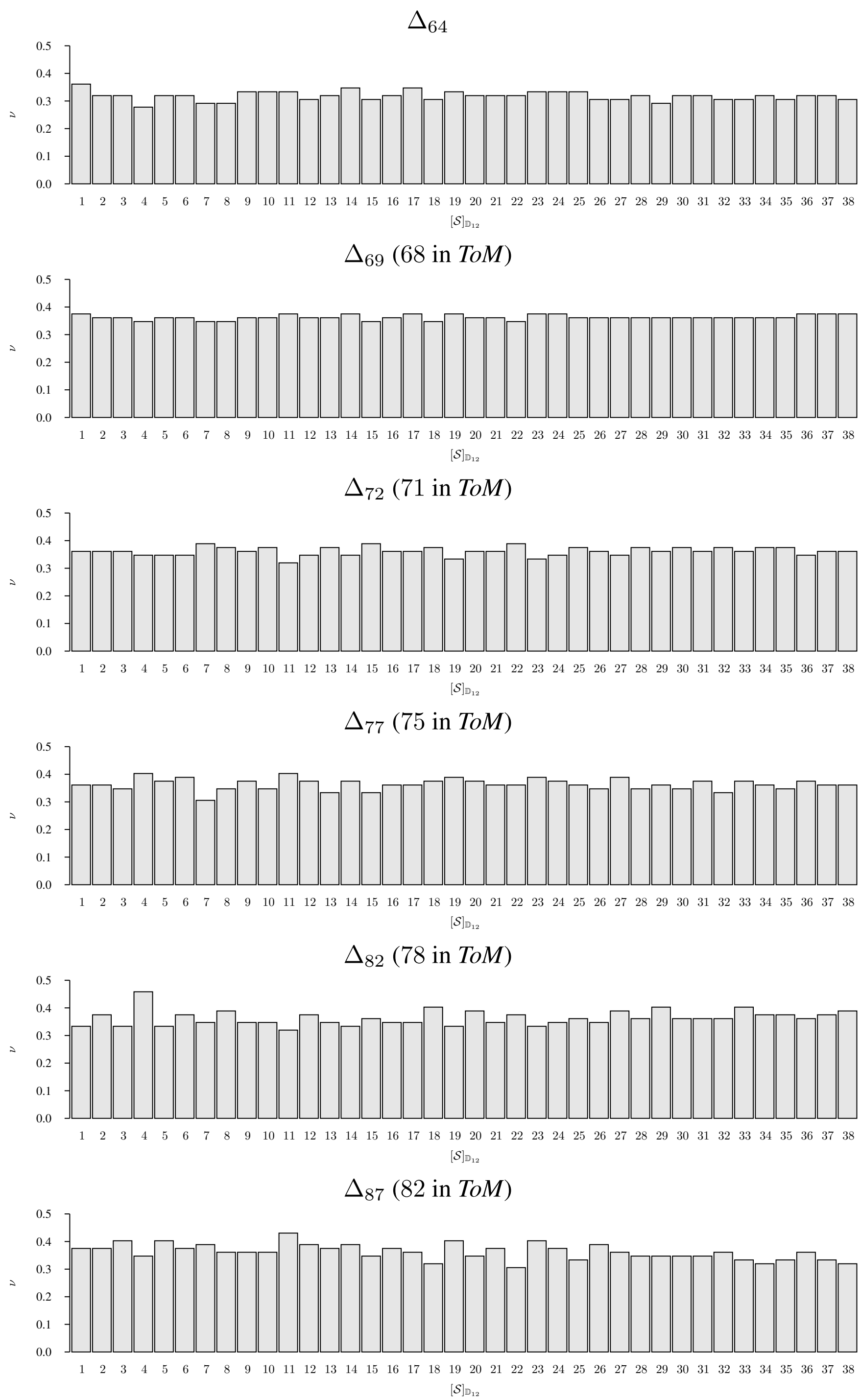


Figure D.6: The consonance compatibility degree ν of the six counterpoint worlds in $\mathcal{C}_{12}$ versus the 38 dihedral classes of heptatonic scales, for *sweeping* orientations only. The same values hold for *hanging* orientations only.

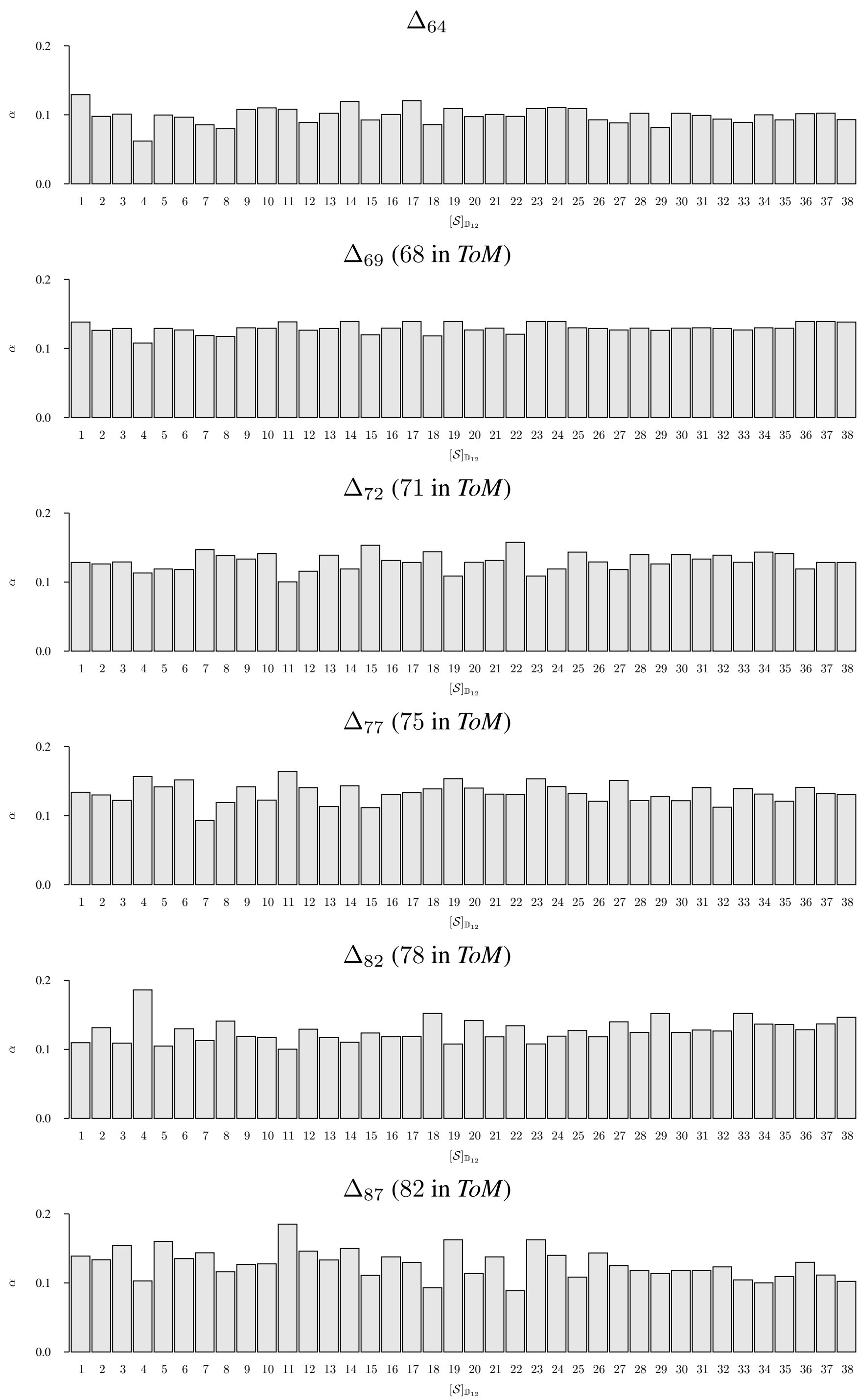


Figure D.7: The allowed step compatibility degree α of the six counterpoint worlds in $\mathcal{C}_{12}$ versus the 38 dihedral classes of heptatonic scales.

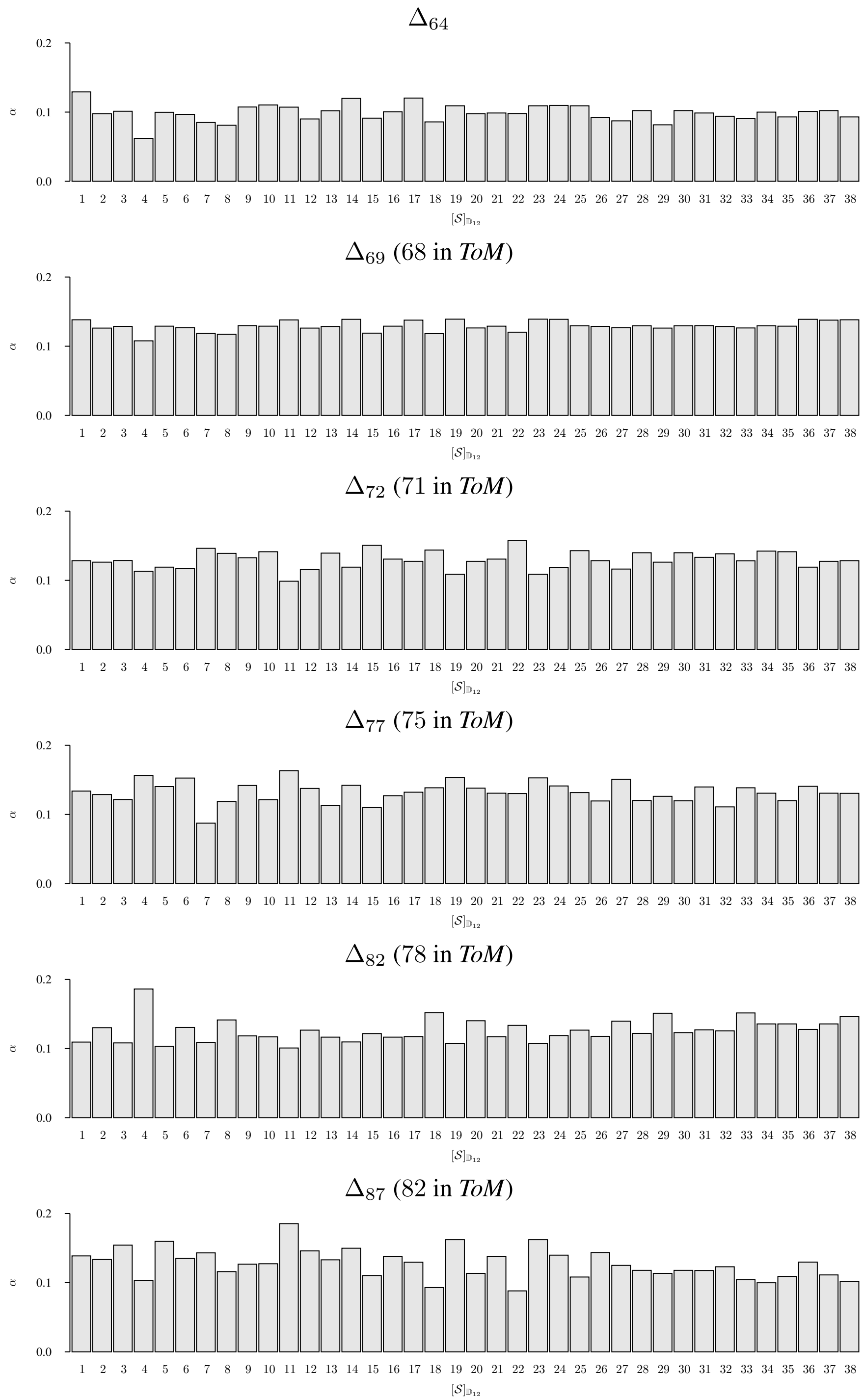


Figure D.8: The allowed step compatibility degree α of the six counterpoint worlds in $\mathcal{C}_{12}$ versus the 38 dihedral classes of heptatonic scales, for *sweeping* orientations only. The same values hold for *hanging* orientations only.

Table D.1: Correspondence between affine index classes $[\mathcal{S}]_{\mathbb{A}_{12}}$ used in *ToM* and dihedral class indices $[\mathcal{S}]_{\mathbb{D}_{12}}$ used in Tab.D.1 to D.8.

$[\mathcal{S}]_{\mathbb{D}_{12}}$	$[\mathcal{S}]_{\mathbb{A}_{12}}$
1	38
2	47
3	54
4	62
5	61
6	45
7	58
8	59
9	42
10	40
11	60
12	56
13	53
14	48
15	57
16	49
17	39
18	52
19	50
20	43
21	49
22	51
24	46
25	41
26	54
27	45
28	55
29	47
30	44
31	42
32	53
33	43
34	41
35	40
36	48
37	39
38	38

D.2 Forbidden Successors

A list of forbidden steps associated with different movements of the cantus firmus is given for a choice of counterpoint worlds restricted to the DUR and Fux dichotomies, in association with the diatonic scale in the $n = 12$ context.

Table D.2: Forbidden successors for the DUR world Δ_{64} ($K = \{2,4,5,7,9,11\}$) and diatonic scale $\mathcal{S}_1 = \{0,2,4,5,7,9,11\}$. Cantus firmus x_i, consonance k_i and discantus z_i. A plus sign denotes the sweeping orientation, a minus sign the hanging orientation. Steps are made between the start interval $i = 0$ (columns) and the end interval $i = 1$ (rows), cross mark denotes a forbidden step. All numbers are residual classes modulo 12.

1. Oblique Motion in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 0$

CF: $D \mapsto D$

k_1 z_1 \ k_0 z_0	2 E_+,C_-	5 G_+,A_-	7 A_+,G_-	9 B_+,F_-
2 E_+,C_-	✖		✖	
5 G_+,A_-		✖		
7 A_+,G_-			✖	
9 B_+,F_-			✖	✖

CF: $E \mapsto E$

k_1 z_1 \ k_0 z_0	2 D_-	4 C_-	5 A_+,B_-	7 B_+,A_-	9 G_-	11 F_-
2 D_-	✖	✖		✖		
4 C_-		✖		✖		
5 A_+,B_-			✖			
7 B_+,A_-		✖		✖		
9 G_-		✖		✖	✖	
11 F_-			✖			✖

CF: $F \mapsto F$

k_1 z_1 \ k_0 z_0	2 G_+	4 A_+	5 C_-	7 C_+	9 D_+	11 E_+
2 G_+	✖	✖		✖		
4 A_+		✖		✖		
5 C_-			✖			
7 C_+		✖		✖		
9 D_+		✖		✖	✖	
11 E_+			✖			✖

CF: $G \mapsto G$

k_1 z_1 \ k_0 z_0	2 A_+,F_-	4 B_+	5 C_+,D_-	7 D_+,C_-	9 E_+
2 A_+,F_-	✖	✖		✖	
4 B_+		✖		✖	
5 C_+,D_-			✖		
7 D_+,C_-		✖		✖	
9 E_+		✖		✖	✖

CF: $A \mapsto A$

k_1 z_1 \ k_0 z_0	2 B_+,G_-	4 F_-	5 D_+,E_-	7 E_+,D_-	9 C_-
2 B_+,G_-	✖	✖		✖	
4 F_-		✖		✖	
5 D_+,E_-			✖		
7 E_+,D_-		✖		✖	
9 C_-		✖		✖	✖

CF: $B \mapsto B$

k_1 z_1 \ k_0 z_0	2 A_-	4 G_-	5 E_+	7 E_-	9 D_-	11 C_-
2 A_-	✖	✖		✖		
4 G_-		✖		✖		
5 E_+			✖			
7 E_-		✖		✖		
9 D_-		✖		✖	✖	
11 C_-			✖			✖

CF: $C \mapsto C$

k_1 z_1 \ k_0 z_0	2 D_+	4 E_+	5 F_+,G_-	7 G_+,F_-	9 A_+	11 B_+
2 D_+	✖	✖		✖		
4 E_+		✖		✖		
5 F_+,G_-			✖			
7 G_+,F_-		✖		✖		
9 A_+		✖		✖	✖	
11 B_+			✖			✖

2. Minor Ascending Second in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 1$

CF: $E \mapsto F$

k_1 z_1 \ k_0 z_0	2 D_-	4 C_-	5 A_+,B_-	7 B_+,A_-	9 G_-	11 F_-
2 G_+						
4 A_+						
5 C_-			✖			
7 C_+						
9 D_+						
11 E_+			✖			✖

CF: $B \mapsto C$

k_1 z_1 \ k_0 z_0	2 A_-	4 G_-	5 E_+	7 E_-	9 D_-	11 C_-
2 D_+						
4 E_+						
5 F_+,G_-			✖			
7 G_+,F_-						
9 A_+						
11 B_+			✖			✖

3. Minor Descending Second in Cantus Firmus: $x_0 \mapsto x_1 = x_0 - 1$

CF: $F \mapsto E$

k_1 z_1 \ k_0 z_0	2 G_+	4 A_+	5 C_-	7 C_+	9 D_+	11 E_+
2 D_-						
4 C_-						
5 A_+,B_-			✖			
7 B_+,A_-						
9 G_-						
11 F_-			✖			✖

CF: $C \mapsto B$

k_1 z_1 \ k_0 z_0	2 D_+	4 E_+	5 F_+,G_-	7 G_+,F_-	9 A_+	11 B_+
2 A_-						
4 G_-						
5 E_+			✖			
7 E_-						
9 D_-						
11 C_-			✖			✖

4. Major Ascending Second in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 2$

CF: $D \mapsto E$

k_1 z_1 \ k_0 z_0	2 E_+,C_-	5 G_+,A_-	7 A_+,G_-	9 B_+,F_-
2 D_-	✖		✖	
4 C_-			✖	
5 A_+,B_-		✖		
7 B_+,A_-			✖	
9 G_-			✖	✖
11 F_-		✖		

CF: $F \mapsto G$

k_1 z_1 \ k_0 z_0	2 G_+	4 A_+	5 C_-	7 C_+	9 D_+	11 E_+
2 A_+,F_-	✖	✖		✖		
4 B_+		✖		✖		
5 C_+,D_-			✖			
7 D_+,C_-		✖		✖		
9 E_+		✖		✖	✖	

CF: $G \mapsto A$

k_1 z_1 \ k_0 z_0	2 A_+,F_-	4 B_+	5 C_+,D_-	7 D_+,C_-	9 E_+
2 B_+,G_-	✖	✖		✖	
4 F_-		✖		✖	
5 D_+,E_-			✖		
7 E_+,D_-		✖		✖	
9 C_-		✖		✖	✖

CF: $A \mapsto B$

k_1 z_1 \ k_0 z_0	2 B_+,G_-	4 F_-	5 D_+,E_-	7 E_+,D_-	9 C_-
2 A_-	✖	✖		✖	
4 G_-		✖		✖	
5 E_+			✖		
7 E_-		✖		✖	
9 D_-		✖		✖	✖
11 C_-			✖		

CF: $C \mapsto D$

k_1 z_1 \ k_0 z_0	2 D_+	4 E_+	5 F_+,G_-	7 G_+,F_-	9 A_+	11 B_+
2 E_+,C_-	✖	✖		✖		
5 G_+,A_-			✖			
7 A_+,G_-		✖		✖		
9 B_+,F_-		✖		✖	✖	

5. Major Descending Second in Cantus Firmus: $x_0 \mapsto x_1 = x_0 - 2$

CF: $D \mapsto C$

k_1 z_1 \ k_0 z_0	2 E_+,C_-	5 G_+,A_-	7 A_+,G_-	9 B_+,F_-
2 D_+	✖		✖	
4 E_+			✖	
5 F_+,G_-		✖		
7 G_+,F_-			✖	
9 A_+			✖	✖
11 B_+		✖		

CF: $E \mapsto D$

k_1 z_1 \ k_0 z_0	2 D_-	4 C_-	5 A_+,B_-	7 B_+,A_-	9 G_-	11 F_-
2 E_+,C_-	✖	✖		✖		
5 G_+,A_-			✖			
7 A_+,G_-		✖		✖		
9 B_+,F_-		✖		✖	✖	

CF: $G \mapsto F$

k_1 z_1 \ k_0 z_0	2 A_+,F_-	4 B_+	5 C_+,D_-	7 D_+,C_-	9 E_+
2 G_+	✖	✖		✖	
4 A_+		✖		✖	
5 C_-			✖		
7 C_+		✖		✖	
9 D_+		✖		✖	✖
11 E_+			✖		

CF: $A \mapsto G$

k_1 z_1 \ k_0 z_0	2 B_+,G_-	4 F_-	5 D_+,E_-	7 E_+,D_-	9 C_-
2 A_+,F_-	✖	✖		✖	
4 B_+		✖		✖	
5 C_+,D_-			✖		
7 D_+,C_-		✖		✖	
9 E_+		✖		✖	✖

CF: $B \mapsto A$

k_1 z_1 \ k_0 z_0	2 A_-	4 G_-	5 E_+	7 E_-	9 D_-	11 C_-
2 B_+,G_-	✖	✖		✖		
4 F_-		✖		✖		
5 D_+,E_-			✖			
7 E_+,D_-		✖		✖		
9 C_-		✖		✖	✖	

6. Minor Ascending Third in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 3$

CF: $D \mapsto F$

k_1 z_1 \ k_0 z_0	2 E_+,C_-	5 G_+,A_-	7 A_+,G_-	9 B_+,F_-
2 G_+				
4 A_+				
5 C_-		✖		
7 C_+				
9 D_+				
11 E_+		✖		

CF: $E \mapsto G$

k_1 z_1 \ k_0 z_0	2 D_-	4 C_-	5 A_+,B_-	7 B_+,A_-	9 G_-	11 F_-
2 A_+,F_-						
4 B_+						
5 C_+,D_-			✖			
7 D_+,C_-						
9 E_+						

CF: $A \mapsto C$

k_1 z_1 \ k_0 z_0	2 B_+,G_-	4 F_-	5 D_+,E_-	7 E_+,D_-	9 C_-
2 D_+					
4 E_+					
5 F_+,G_-			✖		
7 G_+,F_-					
9 A_+					
11 B_+			✖		

CF: $B \mapsto D$

k_1 z_1 \ k_0 z_0	2 A_-	4 G_-	5 E_+	7 E_-	9 D_-	11 C_-
2 E_+,C_-						
5 G_+,A_-			✖			
7 A_+,G_-						
9 B_+,F_-						

Table D.3: Forbidden successors for the DUR world Δ_{64} ($K = \{2, 4, 5, 7, 9, 11\}$) and diatonic scale $\mathcal{S}_1 = \{0, 2, 4, 5, 7, 9, 11\}$, continued. Cantus firmus x_i, consonance k_i and discantus z_i. A plus sign denotes the sweeping orientation, a minus sign the hanging orientation. Steps are made between the start interval $i = 0$ (columns) and the end interval $i = 1$ (rows), cross mark denotes a forbidden step. All numbers are residual classes modulo 12.

7. Minor Descending Third in Cantus Firmus: $x_0 \mapsto x_1 = x_0 - 3$

CF: $D \mapsto B$

k_1	z_1 \ k_0	2	5	7	9
	z_0	E_+, C_-	G_+, A_-	A_+, G_-	B_+, F_-
2	A_-				
4	G_-				
5	E_+		✖		
7	E_-				
9	D_-				
11	C_-		✖		

CF: $F \mapsto D$

k_1	z_1 \ k_0	2	4	5	7	9	11
	z_0	G_+	A_+	C_-	C_+	D_+	E_+
2	E_+, C_-						
5	G_+, A_-			✖			
7	A_+, G_-						
9	B_+, F_-						

CF: $G \mapsto E$

k_1	z_1 \ k_0	2	4	5	7	9
	z_0	A_+, F_-	B_+	C_+, D_-	D_+, C_-	E_+
2	D_-					
4	C_-					
5	A_+, B_-			✖		
7	B_+, A_-					
9	G_-					
11	F_-			✖		

CF: $C \mapsto A$

k_1	z_1 \ k_0	2	4	5	7	9	11
	z_0	D_+	E_+	F_+, G_-	G_+, F_-	A_+	B_+
2	B_+, G_-						
4	F_-						
5	D_+, E_-			✖			
7	E_+, D_-						
9	C_-						

8. Major Ascending Third in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 4$

CF: $F \mapsto A$

k_1	z_1 \ k_0	2	4	5	7	9	11
	z_0	G_+	A_+	C_-	C_+	D_+	E_+
2	B_+, G_-	✖	✖		✖		
4	F_-		✖		✖		
5	D_+, E_-			✖			
7	E_+, D_-		✖		✖		
9	C_-		✖		✖	✖	

CF: $G \mapsto B$

k_1	z_1 \ k_0	2	4	5	7	9
	z_0	A_+, F_-	B_+	C_+, D_-	D_+, C_-	E_+
2	A_-	✖	✖		✖	
4	G_-		✖		✖	
5	E_+			✖		
7	E_-		✖		✖	
9	D_-		✖		✖	✖
11	C_-			✖		

CF: $C \mapsto E$

k_1	z_1 \ k_0	2	4	5	7	9	11
	z_0	D_+	E_+	F_+, G_-	G_+, F_-	A_+	B_+
2	D_-	✖	✖		✖		
4	C_-		✖		✖		
5	A_+, B_-			✖			
7	B_+, A_-		✖		✖		
9	G_-		✖		✖	✖	
11	F_-			✖			✖

9. Major Descending Third in Cantus Firmus: $x_0 \mapsto x_1 = x_0 - 4$

CF: $E \mapsto C$

k_1	z_1 \ k_0	2	4	5	7	9	11
	z_0	D_-	C_-	A_+, B_-	B_+, A_-	G_-	F_-
2	D_+	✖	✖		✖		
4	E_+		✖		✖		
5	F_+, G_-			✖			
7	G_+, F_-		✖		✖		
9	A_+		✖		✖	✖	
11	B_+			✖			✖

CF: $A \mapsto F$

k_1	z_1 \ k_0	2	4	5	7	9
	z_0	B_+, G_-	F_-	D_+, E_-	E_+, D_-	C_-
2	G_+	✖	✖		✖	
4	A_+		✖		✖	
5	C_-			✖		
7	C_+		✖		✖	
9	D_+		✖		✖	✖
11	E_+			✖		

CF: $B \mapsto G$

k_1	z_1 \ k_0	2	4	5	7	9	11
	z_0	A_-	G_-	E_+	E_-	D_-	C_-
2	A_+, F_-	✖	✖		✖		
4	B_+		✖		✖		
5	C_+, D_-			✖			
7	D_+, C_-		✖		✖		
9	E_+		✖		✖	✖	

10. Ascending Fourth in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 5$

CF: $D \mapsto G$

k_1	z_1 \ k_0	2	5	7	9
	z_0	E_+, C_-	G_+, A_-	A_+, G_-	B_+, F_-
2	A_+, F_-				
4	B_+				
5	C_+, D_-		✖		
7	D_+, C_-				
9	E_+				

CF: $E \mapsto A$

k_1	z_1 \ k_0	2	4	5	7	9	11
	z_0	D_-	C_-	A_+, B_-	B_+, A_-	G_-	F_-
2	B_+, G_-						
4	F_-						
5	D_+, E_-			✖			
7	E_+, D_-						
9	C_-						

CF: $G \mapsto C$

k_1	z_1 \ k_0	2	4	5	7	9
	z_0	A_+, F_-	B_+	C_+, D_-	D_+, C_-	E_+
2	D_+					
4	E_+					
5	F_+, G_-			✖		
7	G_+, F_-					
9	A_+					
11	B_+			✖		

CF: $A \mapsto D$

k_1	z_1 \ k_0	2	4	5	7	9
	z_0	B_+, G_-	F_-	D_+, E_-	E_+, D_-	C_-
2	E_+, C_-					
5	G_+, A_-			✖		
7	A_+, G_-					
9	B_+, F_-					

CF: $B \mapsto E$

k_1	z_1 \ k_0	2	4	5	7	9	11
	z_0	A_-	G_-	E_+	E_-	D_-	C_-
2	D_-						
4	C_-						
5	A_+, B_-			✖			
7	B_+, A_-						
9	G_-						
11	F_-			✖			✖

CF: $C \mapsto F$

k_1	z_1 \ k_0	2	4	5	7	9	11
	z_0	D_+	E_+	F_+, G_-	G_+, F_-	A_+	B_+
2	G_+						
4	A_+						
5	C_-			✖			
7	C_+						
9	D_+						
11	E_+			✖			✖

11. Descending Fourth in Cantus Firmus: $x_0 \mapsto x_1 = x_0 - 5$

CF: $D \mapsto A$

k_1	z_1 \ k_0	2	5	7	9
	z_0	E_+, C_-	G_+, A_-	A_+, G_-	B_+, F_-
2	B_+, G_-				
4	F_-				
5	D_+, E_-		✖		
7	E_+, D_-				
9	C_-				

CF: $E \mapsto B$

k_1	z_1 \ k_0	2	4	5	7	9	11
	z_0	D_-	C_-	A_+, B_-	B_+, A_-	G_-	F_-
2	A_-						
4	G_-						
5	E_+			✖			
7	E_-						
9	D_-						
11	C_-			✖			✖

CF: $F \mapsto C$

k_1	z_1 \ k_0	2	4	5	7	9	11
	z_0	G_+	A_+	C_-	C_+	D_+	E_+
2	D_+						
4	E_+						
5	F_+, G_-			✖			
7	G_+, F_-						
9	A_+						
11	B_+			✖			✖

CF: $G \mapsto D$

k_1	z_1 \ k_0	2	4	5	7	9
	z_0	A_+, F_-	B_+	C_+, D_-	D_+, C_-	E_+
2	E_+, C_-					
5	G_+, A_-			✖		
7	A_+, G_-					
9	B_+, F_-					

CF: $A \mapsto E$

k_1	z_1 \ k_0	2	4	5	7	9
	z_0	B_+, G_-	F_-	D_+, E_-	E_+, D_-	C_-
2	D_-					
4	C_-					
5	A_+, B_-			✖		
7	B_+, A_-					
9	G_-					
11	F_-			✖		

CF: $C \mapsto G$

k_1	z_1 \ k_0	2	4	5	7	9	11
	z_0	D_+	E_+	F_+, G_-	G_+, F_-	A_+	B_+
2	A_+, F_-						
4	B_+						
5	C_+, D_-			✖			
7	D_+, C_-						
9	E_+						

12. Ascending Tritone in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 6$

CF: $F \mapsto B$

k_1	z_1 \ k_0	2	4	5	7	9	11
	z_0	G_+	A_+	C_-	C_+	D_+	E_+
2	A_-	✖	✖		✖		
4	G_-		✖		✖		
5	E_+			✖			
7	E_-		✖		✖		
9	D_-		✖		✖	✖	
11	C_-			✖			✖

CF: $B \mapsto F$

k_1	z_1 \ k_0	2	4	5	7	9	11
	z_0	A_-	G_-	E_+	E_-	D_-	C_-
2	G_+	✖	✖		✖		
4	A_+		✖		✖		
5	C_-			✖			
7	C_+		✖		✖		
9	D_+		✖		✖	✖	
11	E_+			✖			✖

Table D.4: Forbidden successors for the FUX world Δ_{87} ($K = \{3, 4, 7, 8, 9\}$) and diatonic scale $\mathcal{S}_1 = \{0, 2, 4, 5, 7, 9, 11\}$. Cantus firmus x_i, consonance k_i and discantus z_i. A plus sign denotes the sweeping orientation, a minus sign the hanging orientation. Steps are made between the start interval $i = 0$ (columns) and the end interval $i = 1$ (rows), cross mark denotes a forbidden step. All numbers are residual classes modulo 12. This table is the complement to Tab. O.2 on pages 1218-1219 in *ToM*), showing the allowed successors.

1. Oblique Motion in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 0$

CF: $D \mapsto D$

k_1 \ k_0	z_1 \ z_0	0 D	3 F_+, B_-	7 A_+, G_-	9 B_+, F_-
0	D	✖			
3	F_+, B_-		✖		✖
7	A_+, G_-			✖	
9	B_+, F_-		✖		✖

CF: $E \mapsto E$

k_1 \ k_0	z_1 \ z_0	0 E	3 G_+	4 C_-	7 B_+, A_-	8 C_+	9 G_-
0	E	✖		✖			
3	G_+		✖				✖
4	C_-			✖			
7	B_+, A_-				✖		
8	C_+			✖		✖	
9	G_-		✖				✖

CF: $F \mapsto F$

k_1 \ k_0	z_1 \ z_0	0 F	3 D_-	4 A_+	7 C_+	8 A_-	9 D_+
0	F	✖		✖			
3	D_-		✖				✖
4	A_+			✖			
7	C_+				✖		
8	A_-			✖		✖	
9	D_+		✖				✖

CF: $G \mapsto G$

k_1 \ k_0	z_1 \ z_0	0 G	3 E_-	4 B_+	7 D_+, C_-	8 B_-	9 E_+
0	G	✖		✖			
3	E_-		✖				✖
4	B_+			✖			
7	D_+, C_-				✖		
8	B_-			✖		✖	
9	E_+		✖				✖

CF: $A \mapsto A$

k_1 \ k_0	z_1 \ z_0	0 A	3 C_+	4 F_-	7 E_+, D_-	8 F_+	9 C_-
0	A	✖		✖			
3	C_+		✖				✖
4	F_-			✖			
7	E_+, D_-				✖		
8	F_+			✖		✖	
9	C_-		✖				✖

CF: $B \mapsto B$

k_1 \ k_0	z_1 \ z_0	0 B	3 D_+	4 G_-	7 E_-	8 G_+	9 D_-
0	B	✖		✖			
3	D_+		✖				✖
4	G_-			✖			
7	E_-				✖		
8	G_+			✖		✖	
9	D_-		✖				✖

CF: $C \mapsto C$

k_1 \ k_0	z_1 \ z_0	0 C	3 A_-	4 E_+	7 G_+, F_-	8 E_-	9 A_+
0	C	✖		✖			
3	A_-		✖				✖
4	E_+			✖			
7	G_+, F_-				✖		
8	E_-			✖		✖	
9	A_+		✖				✖

2. Minor Ascending Second in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 1$

CF: $E \mapsto F$

k_1 \ k_0	z_1 \ z_0	0 E	3 G_+	4 C_-	7 B_+, A_-	8 C_+	9 G_-
0	F						
3	D_-						
4	A_+						
7	C_+				✖		
8	A_-						
9	D_+						

CF: $B \mapsto C$

k_1 \ k_0	z_1 \ z_0	0 B	3 D_+	4 G_-	7 E_-	8 G_+	9 D_-
0	C						
3	A_-						
4	E_+						
7	G_+, F_-				✖		
8	E_-						
9	A_+						

3. Minor Descending Second in Cantus Firmus: $x_0 \mapsto x_1 = x_0 - 1$

CF: $F \mapsto E$

k_1 \ k_0	z_1 \ z_0	0 F	3 D_-	4 A_+	7 C_+	8 A_-	9 D_+
0	E						
3	G_+						
4	C_-						
7	B_+, A_-				✖		
8	C_+						
9	G_-						

CF: $C \mapsto B$

k_1 \ k_0	z_1 \ z_0	0 C	3 A_-	4 E_+	7 G_+, F_-	8 E_-	9 A_+
0	B						
3	D_+						
4	G_-						
7	E_-				✖		
8	G_+						
9	D_-						

4. Major Ascending Second in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 2$

CF: $D \mapsto E$

k_1 \ k_0	z_1 \ z_0	0 D	3 F_+, B_-	7 A_+, G_-	9 B_+, F_-
0	E				
3	G_+				
4	C_-				
7	B_+, A_-			✖	
8	C_+				
9	G_-				

CF: $F \mapsto G$

k_1 \ k_0	z_1 \ z_0	0 F	3 D_-	4 A_+	7 C_+	8 A_-	9 D_+
0	G			✖			
3	E_-						
4	B_+			✖			
7	D_+, C_-				✖		
8	B_-			✖			
9	E_+						

CF: $G \mapsto A$

k_1 \ k_0	z_1 \ z_0	0 G	3 E_-	4 B_+	7 D_+, C_-	8 B_-	9 E_+
0	A			✖			
3	C_+						
4	F_-			✖			
7	E_+, D_-				✖		
8	F_+			✖			
9	C_-						

CF: $A \mapsto B$

k_1 \ k_0	z_1 \ z_0	0 A	3 C_+	4 F_-	7 E_+, D_-	8 F_+	9 C_-
0	B			✖			
3	D_+						
4	G_-			✖			
7	E_-				✖		
8	G_+			✖			
9	D_-						

CF: $C \mapsto D$

k_1 \ k_0	z_1 \ z_0	0 C	3 A_-	4 E_+	7 G_+, F_-	8 E_-	9 A_+
0	D			✖			
3	F_+, B_-						
7	A_+, G_-				✖		
9	B_+, F_-						

5. Major Descending Second in Cantus Firmus: $x_0 \mapsto x_1 = x_0 - 2$

CF: $D \mapsto C$

k_1 \ k_0	z_1 \ z_0	0 D	3 F_+, B_-	7 A_+, G_-	9 B_+, F_-
0	C				
3	A_-				
4	E_+				
7	G_+, F_-			✖	
8	E_-				
9	A_+				

CF: $E \mapsto D$

k_1 \ k_0	z_1 \ z_0	0 E	3 G_+	4 C_-	7 B_+, A_-	8 C_+	9 G_-
0	D			✖			
3	F_+, B_-						
7	A_+, G_-				✖		
9	B_+, F_-						

CF: $G \mapsto F$

k_1 \ k_0	z_1 \ z_0	0 G	3 E_-	4 B_+	7 D_+, C_-	8 B_-	9 E_+
0	F			✖			
3	D_-						
4	A_+			✖			
7	C_+				✖		
8	A_-			✖			
9	D_+						

CF: $A \mapsto G$

k_1 \ k_0	z_1 \ z_0	0 A	3 C_+	4 F_-	7 E_+, D_-	8 F_+	9 C_-
0	G			✖			
3	E_-						
4	B_+			✖			
7	D_+, C_-				✖		
8	B_-			✖			
9	E_+						

CF: $B \mapsto A$

k_1 \ k_0	z_1 \ z_0	0 B	3 D_+	4 G_-	7 E_-	8 G_+	9 D_-
0	A			✖			
3	C_+						
4	F_-			✖			
7	E_+, D_-				✖		
8	F_+			✖			
9	C_-						

6. Minor Ascending Third in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 3$

CF: $D \mapsto F$

k_1 \ k_0	z_1 \ z_0	0 D	3 F_+, B_-	7 A_+, G_-	9 B_+, F_-
0	F				
3	D_-		✖		✖
4	A_+				
7	C_+			✖	
8	A_-				
9	D_+		✖		✖

CF: $E \mapsto G$

k_1 \ k_0	z_1 \ z_0	0 E	3 G_+	4 C_-	7 B_+, A_-	8 C_+	9 G_-
0	G						
3	E_-		✖				✖
4	B_+						
7	D_+, C_-				✖		
8	B_-						
9	E_+		✖				✖

CF: $A \mapsto C$

k_1 \ k_0	z_1 \ z_0	0 A	3 C_+	4 F_-	7 E_+, D_-	8 F_+	9 C_-
0	C						
3	A_-		✖				✖
4	E_+						
7	G_+, F_-				✖		
8	E_-						
9	A_+		✖				✖

CF: $B \mapsto D$

k_1 \ k_0	z_1 \ z_0	0 B	3 D_+	4 G_-	7 E_-	8 G_+	9 D_-
0	D						
3	F_+, B_-		✖				✖
7	A_+, G_-				✖		
9	B_+, F_-		✖				✖

Table D.5: Forbidden successors for the FUX world Δ_{87} ($K = \{3,4,7,8,9\}$) and diatonic scale $\mathcal{S}_1 = \{0,2,4,5,7,9,11\}$, continued. Cantus firmus x_i, consonance k_i and discantus z_i. A plus sign denotes the sweeping orientation, a minus sign the hanging orientation. Steps are made between the start interval $i = 0$ (columns) and the end interval $i = 1$ (rows), cross mark denotes a forbidden step. All numbers are residual classes modulo 12. This table is the complement to Tab. O.2 on pages 1218-1219 in *ToM*), showing the allowed successors.

7. Minor Descending Third in Cantus Firmus: $x_0 \mapsto x_1 = x_0 - 3$

CF: $D \mapsto B$

k_1	z_1 \ k_0	0	3	7	9
	z_0	D	F_+, B_-	A_+, G_-	B_+, F_-
0	B				
3	D_+		✖		✖
4	G_-				
7	E_-			✖	
8	G_+				
9	D_-		✖		✖

CF: $F \mapsto D$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	F	D_-	A_+	C_+	A_-	D_+
0	D						
3	F_+, B_-		✖				✖
7	A_+, G_-				✖		
9	B_+, F_-		✖				✖

CF: $G \mapsto E$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	G	E_-	B_+	D_+, C_-	B_-	E_+
0	E						
3	G_+		✖				✖
4	C_-						
7	B_+, A_-				✖		
8	C_+						
9	G_-		✖				✖

CF: $C \mapsto A$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	C	A_-	E_+	G_+, F_-	E_-	A_+
0	A						
3	C_+		✖				✖
4	F_-						
7	E_+, D_-				✖		
8	F_+						
9	C_-		✖				✖

8. Major Ascending Third in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 4$

CF: $F \mapsto A$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	F	D_-	A_+	C_+	A_-	D_+
0	A			✖			
3	C_+						
4	F_-			✖			
7	E_+, D_-				✖		
8	F_+			✖			
9	C_-						

CF: $G \mapsto B$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	G	E_-	B_+	D_+, C_-	B_-	E_+
0	B			✖			
3	D_+						
4	G_-			✖			
7	E_-				✖		
8	G_+			✖			
9	D_-						

CF: $C \mapsto E$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	C	A_-	E_+	G_+, F_-	E_-	A_+
0	E			✖			
3	G_+						
4	C_-			✖			
7	B_+, A_-				✖		
8	C_+			✖			
9	G_-						

9. Major Descending Third in Cantus Firmus: $x_0 \mapsto x_1 = x_0 - 4$

CF: $E \mapsto C$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	E	G_+	C_-	B_+, A_-	C_+	G_-
0	C			✖			
3	A_-						
4	E_+			✖			
7	G_+, F_-				✖		
8	E_-			✖			
9	A_+						

CF: $A \mapsto F$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	A	C_+	F_-	E_+, D_-	F_+	C_-
0	F			✖			
3	D_-						
4	A_+			✖			
7	C_+				✖		
8	A_-			✖			
9	D_+						

CF: $B \mapsto G$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	B	D_+	G_-	E_-	G_+	D_-
0	G			✖			
3	E_-						
4	B_+			✖			
7	D_+, C_-				✖		
8	B_-			✖			
9	E_+						

10. Ascending Fourth in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 5$

CF: $D \mapsto G$

k_1	z_1 \ k_0	0	3	7	9
	z_0	D	F_+, B_-	A_+, G_-	B_+, F_-
0	G				
3	E_-				
4	B_+				
7	D_+, C_-			✖	
8	B_-				
9	E_+				

CF: $E \mapsto A$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	E	G_+	C_-	B_+, A_-	C_+	G_-
0	A						
3	C_+						
4	F_-						
7	E_+, D_-				✖		
8	F_+						
9	C_-						

CF: $G \mapsto C$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	G	E_-	B_+	D_+, C_-	B_-	E_+
0	C						
3	A_-						
4	E_+						
7	G_+, F_-				✖		
8	E_-						
9	A_+						

CF: $A \mapsto D$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	A	C_+	F_-	E_+, D_-	F_+	C_-
0	D						
3	F_+, B_-						
7	A_+, G_-				✖		
9	B_+, F_-						

CF: $B \mapsto E$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	B	D_+	G_-	E_-	G_+	D_-
0	E						
3	G_+						
4	C_-						
7	B_+, A_-				✖		
8	C_+						
9	G_-						

CF: $C \mapsto F$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	C	A_-	E_+	G_+, F_-	E_-	A_+
0	F						
3	D_-						
4	A_+						
7	C_+				✖		
8	A_-						
9	D_+						

11. Descending Fourth in Cantus Firmus: $x_0 \mapsto x_1 = x_0 - 5$

CF: $D \mapsto A$

k_1	z_1 \ k_0	0	3	7	9
	z_0	D	F_+, B_-	A_+, G_-	B_+, F_-
0	A				
3	C_+				
4	F_-				
7	E_+, D_-			✖	
8	F_+				
9	C_-				

CF: $E \mapsto B$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	E	G_+	C_-	B_+, A_-	C_+	G_-
0	B						
3	D_+						
4	G_-						
7	E_-				✖		
8	G_+						
9	D_-						

CF: $F \mapsto C$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	F	D_-	A_+	C_+	A_-	D_+
0	C						
3	A_-						
4	E_+						
7	G_+, F_-				✖		
8	E_-						
9	A_+						

CF: $G \mapsto D$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	G	E_-	B_+	D_+, C_-	B_-	E_+
0	D						
3	F_+, B_-						
7	A_+, G_-				✖		
9	B_+, F_-						

CF: $A \mapsto E$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	A	C_+	F_-	E_+, D_-	F_+	C_-
0	E						
3	G_+						
4	C_-						
7	B_+, A_-				✖		
8	C_+						
9	G_-						

CF: $C \mapsto G$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	C	A_-	E_+	G_+, F_-	E_-	A_+
0	G						
3	E_-						
4	B_+						
7	D_+, C_-				✖		
8	B_-						
9	E_+						

12. Ascending Tritone in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 6$

CF: $F \mapsto B$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	F	D_-	A_+	C_+	A_-	D_+
0	B	✖		✖			
3	D_+		✖				✖
4	G_-			✖			
7	E_-				✖		
8	G_+			✖		✖	
9	D_-		✖				✖

CF: $B \mapsto F$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	B	D_+	G_-	E_-	G_+	D_-
0	F	✖		✖			
3	D_-		✖				✖
4	A_+			✖			
7	C_+				✖		
8	A_-			✖		✖	
9	D_+		✖				✖

Table D.6: Forbidden successors for the DUR world Δ_{64} ($K = \{2, 4, 5, 7, 9, 11\}$) and pentatonic scale $\mathcal{S}_1 = \{0, 2, 4, 7, 9\}$. Cantus firmus x_i, consonance k_i and discantus z_i. A plus sign denotes the sweeping orientation, a minus sign the hanging orientation. Steps are made between the start interval $i = 0$ (columns) and the end interval $i = 1$ (rows), cross mark denotes a forbidden step. All numbers are residual classes modulo 12.

1. Oblique Motion in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 0$

CF: $D \mapsto D$

k_1	z_1 \ k_0, z_0	2 E_+, C_-	5 G_+, A_-	7 A_+, G_-
2	E_+, C_-	✖		✖
5	G_+, A_-		✖	
7	A_+, G_-			✖

CF: $E \mapsto E$

k_1	z_1 \ k_0, z_0	2 D_-	4 C_-	5 A_+	7 A_-	9 G_-
2	D_-	✖	✖		✖	
4	C_-		✖		✖	
5	A_+			✖		
7	A_-		✖		✖	
9	G_-		✖		✖	✖

CF: $G \mapsto G$

k_1	z_1 \ k_0, z_0	2 A_+	5 C_+, D_-	7 D_+, C_-	9 E_+
2	A_+	✖		✖	
5	C_+, D_-		✖		
7	D_+, C_-			✖	
9	E_+			✖	✖

CF: $A \mapsto A$

k_1	z_1 \ k_0, z_0	2 G_-	5 D_+, E_-	7 E_+, D_-	9 C_-
2	G_-	✖		✖	
5	D_+, E_-		✖		
7	E_+, D_-			✖	
9	C_-			✖	✖

CF: $C \mapsto C$

k_1	z_1 \ k_0, z_0	2 D_+	4 E_+	5 G_-	7 G_+	9 A_+
2	D_+	✖	✖		✖	
4	E_+		✖		✖	
5	G_-			✖		
7	G_+		✖		✖	
9	A_+		✖		✖	✖

2. Minor Ascending Second in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 1$

3. Minor Descending Second in Cantus Firmus: $x_0 \mapsto x_1 = x_0 - 1$

4. Major Ascending Second in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 2$

CF: $D \mapsto E$

k_1	z_1 \ k_0, z_0	2 E_+, C_-	5 G_+, A_-	7 A_+, G_-
2	D_-	✖		✖
4	C_-			✖
5	A_+		✖	
7	A_-			✖
9	G_-			✖

CF: $G \mapsto A$

k_1	z_1 \ k_0, z_0	2 A_+	5 C_+, D_-	7 D_+, C_-	9 E_+
2	G_-	✖		✖	
5	D_+, E_-		✖		
7	E_+, D_-			✖	
9	C_-			✖	✖

CF: $C \mapsto D$

k_1	z_1 \ k_0, z_0	2 D_+	4 E_+	5 G_-	7 G_+	9 A_+
2	E_+, C_-	✖	✖		✖	
5	G_+, A_-			✖		
7	A_+, G_-		✖		✖	

5. Major Descending Second in Cantus Firmus: $x_0 \mapsto x_1 = x_0 - 2$

CF: $D \mapsto C$

k_1	z_1 \ k_0, z_0	2 E_+, C_-	5 G_+, A_-	7 A_+, G_-
2	D_+	✖		✖
4	E_+			✖
5	G_-		✖	
7	G_+			✖
9	A_+			✖

CF: $E \mapsto D$

k_1	z_1 \ k_0, z_0	2 D_-	4 C_-	5 A_+	7 A_-	9 G_-
2	E_+, C_-	✖	✖		✖	
5	G_+, A_-			✖		
7	A_+, G_-		✖		✖	

CF: $A \mapsto G$

k_1	z_1 \ k_0, z_0	2 G_-	5 D_+, E_-	7 E_+, D_-	9 C_-
2	A_+	✖		✖	
5	C_+, D_-		✖		
7	D_+, C_-			✖	
9	E_+			✖	✖

6. Minor Ascending Third in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 3$

CF: $E \mapsto G$

k_1	z_1 \ k_0, z_0	2 D_-	4 C_-	5 A_+	7 A_-	9 G_-
2	A_+					
5	C_+, D_-			✖		
7	D_+, C_-					
9	E_+					

CF: $A \mapsto C$

k_1	z_1 \ k_0, z_0	2 G_-	5 D_+, E_-	7 E_+, D_-	9 C_-
2	D_+				
4	E_+				
5	G_-		✖		
7	G_+				
9	A_+				

Table D.7: Forbidden successors for the DUR world Δ_{64} ($K = \{2, 4, 5, 7, 9, 11\}$) and pentatonic scale $\mathcal{S}_1 = \{0, 2, 4, 7, 9\}$, continued. Cantus firmus x_i, consonance k_i and discantus z_i. A plus sign denotes the sweeping orientation, a minus sign the hanging orientation. Steps are made between the start interval $i = 0$ (columns) and the end interval $i = 1$ (rows), cross mark denotes a forbidden step. All numbers are residual classes modulo 12.

7. Minor Descending Third in Cantus Firmus: $x_0 \mapsto x_1 = x_0 - 3$

CF: $G \mapsto E$

k_1	z_1 \ k_0, z_0	2: A_+	5: C_+, D_-	7: D_+, C_-	9: E_+
2	D_-				
4	C_-				
5	A_+		✖		
7	A_-				
9	G_-				

CF: $C \mapsto A$

k_1	z_1 \ k_0, z_0	2: D_+	4: E_+	5: G_-	7: G_+	9: A_+
2	G_-					
5	D_+, E_-			✖		
7	E_+, D_-					
9	C_-					

8. Major Ascending Third in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 4$

CF: $C \mapsto E$

k_1	z_1 \ k_0, z_0	2: D_+	4: E_+	5: G_-	7: G_+	9: A_+
2	D_-	✖	✖		✖	
4	C_-		✖		✖	
5	A_+			✖		
7	A_-		✖		✖	
9	G_-		✖		✖	✖

9. Major Descending Third in Cantus Firmus: $x_0 \mapsto x_1 = x_0 - 4$

CF: $E \mapsto C$

k_1	z_1 \ k_0, z_0	2: D_-	4: C_-	5: A_+	7: A_-	9: G_-
2	D_+	✖	✖		✖	
4	E_+		✖		✖	
5	G_-			✖		
7	G_+		✖		✖	
9	A_+		✖		✖	✖

10. Ascending Fourth in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 5$

CF: $D \mapsto G$

k_1	z_1 \ k_0, z_0	2: E_+, C_-	5: G_+, A_-	7: A_+, G_-
2	A_+			
5	C_+, D_-		✖	
7	D_+, C_-			
9	E_+			

CF: $E \mapsto A$

k_1	z_1 \ k_0, z_0	2: D_-	4: C_-	5: A_+	7: A_-	9: G_-
2	G_-					
5	D_+, E_-			✖		
7	E_+, D_-					
9	C_-					

CF: $G \mapsto C$

k_1	z_1 \ k_0, z_0	2: A_+	5: C_+, D_-	7: D_+, C_-	9: E_+
2	D_+				
4	E_+				
5	G_-		✖		
7	G_+				
9	A_+				

CF: $A \mapsto D$

k_1	z_1 \ k_0, z_0	2: G_-	5: D_+, E_-	7: E_+, D_-	9: C_-
2	E_+, C_-				
5	G_+, A_-		✖		
7	A_+, G_-				

11. Descending Fourth in Cantus Firmus: $x_0 \mapsto x_1 = x_0 - 5$

CF: $D \mapsto A$

k_1	z_1 \ k_0, z_0	2: E_+, C_-	5: G_+, A_-	7: A_+, G_-
2	G_-			
5	D_+, E_-		✖	
7	E_+, D_-			
9	C_-			

CF: $G \mapsto D$

k_1	z_1 \ k_0, z_0	2: A_+	5: C_+, D_-	7: D_+, C_-	9: E_+
2	E_+, C_-				
5	G_+, A_-		✖		
7	A_+, G_-				

CF: $A \mapsto E$

k_1	z_1 \ k_0, z_0	2: G_-	5: D_+, E_-	7: E_+, D_-	9: C_-
2	D_-				
4	C_-				
5	A_+		✖		
7	A_-				
9	G_-				

CF: $C \mapsto G$

k_1	z_1 \ k_0, z_0	2: D_+	4: E_+	5: G_-	7: G_+	9: A_+
2	A_+					
5	C_+, D_-			✖		
7	D_+, C_-					
9	E_+					

12. Ascending Tritone in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 6$

Table D.8: Forbidden successors for the FUX world Δ_{87} ($K = \{3, 4, 7, 8, 9\}$) and pentatonic scale $\mathcal{S}_1 = \{0, 2, 4, 7, 9\}$. Cantus firmus x_i, consonance k_i and discantus z_i. A plus sign denotes the sweeping orientation, a minus sign the hanging orientation. Steps are made between the start interval $i = 0$ (columns) and the end interval $i = 1$ (rows), cross mark denotes a forbidden step. All numbers are residual classes modulo 12.

1. Oblique Motion in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 0$

CF: $D \mapsto D$

k_1	z_1 \ k_0	0	7
	z_0	D	A_+, G_-
0	D	✖	
7	A_+, G_-		✖

CF: $E \mapsto E$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	E	G_+	C_-	A_-	C_+	G_-
0	E	✖		✖			
3	G_+		✖				✖
4	C_-			✖			
7	A_-				✖		
8	C_+			✖		✖	
9	G_-		✖				✖

CF: $G \mapsto G$

k_1	z_1 \ k_0	0	3	7	9
	z_0	G	E_-	D_+, C_-	E_+
0	G	✖			
3	E_-		✖		✖
7	D_+, C_-			✖	
9	E_+		✖		✖

CF: $A \mapsto A$

k_1	z_1 \ k_0	0	3	7	9
	z_0	A	C_+	E_+, D_-	C_-
0	A	✖			
3	C_+		✖		✖
7	E_+, D_-			✖	
9	C_-		✖		✖

CF: $C \mapsto C$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	C	A_-	E_+	G_+	E_-	A_+
0	C	✖		✖			
3	A_-		✖				✖
4	E_+			✖			
7	G_+				✖		
8	E_-			✖		✖	
9	A_+		✖				✖

2. Minor Ascending Second in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 1$

3. Minor Descending Second in Cantus Firmus: $x_0 \mapsto x_1 = x_0 - 1$

4. Major Ascending Second in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 2$

CF: $D \mapsto E$

k_1	z_1 \ k_0	0	7
	z_0	D	A_+, G_-
0	E		
3	G_+		
4	C_-		
7	A_-		✖
8	C_+		
9	G_-		

CF: $G \mapsto A$

k_1	z_1 \ k_0	0	3	7	9
	z_0	G	E_-	D_+, C_-	E_+
0	A				
3	C_+				
7	E_+, D_-			✖	
9	C_-				

CF: $C \mapsto D$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	C	A_-	E_+	G_+	E_-	A_+
0	D			✖			
7	A_+, G_-				✖		

5. Major Descending Second in Cantus Firmus: $x_0 \mapsto x_1 = x_0 - 2$

CF: $D \mapsto C$

k_1	z_1 \ k_0	0	7
	z_0	D	A_+, G_-
0	C		
3	A_-		
4	E_+		
7	G_+		✖
8	E_-		
9	A_+		

CF: $E \mapsto D$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	E	G_+	C_-	A_-	C_+	G_-
0	D			✖			
7	A_+, G_-				✖		

CF: $A \mapsto G$

k_1	z_1 \ k_0	0	3	7	9
	z_0	A	C_+	E_+, D_-	C_-
0	G				
3	E_-				
7	D_+, C_-			✖	
9	E_+				

6. Minor Ascending Third in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 3$

CF: $E \mapsto G$

k_1	z_1 \ k_0	0	3	4	7	8	9
	z_0	E	G_+	C_-	A_-	C_+	G_-
0	G						
3	E_-		✖				✖
7	D_+, C_-				✖		
9	E_+		✖				✖

CF: $A \mapsto C$

k_1	z_1 \ k_0	0	3	7	9
	z_0	A	C_+	E_+, D_-	C_-
0	C				
3	A_-		✖		✖
4	E_+				
7	G_+			✖	
8	E_-				
9	A_+		✖		✖

Table D.9: Forbidden successors for the FUX world Δ_{87} ($K = \{3,4,7,8,9\}$) and pentatonic scale $\mathcal{S}_1 = \{0,2,4,7,9\}$, continued. Cantus firmus x_i, consonance k_i and discantus z_i. A plus sign denotes the sweeping orientation, a minus sign the hanging orientation. Steps are made between the start interval $i = 0$ (columns) and the end interval $i = 1$ (rows), cross mark denotes a forbidden step. All numbers are residual classes modulo 12.

7. Minor Descending Third in Cantus Firmus: $x_0 \mapsto x_1 = x_0 - 3$

CF: $G \mapsto E$

k_1	z_1 \ k_0 / z_0	0 G	3 E_-	7 D_+, C_-	9 E_+
0	E				
3	G_+		✖		✖
4	C_-				
7	A_-			✖	
8	C_+				
9	G_-		✖		✖

CF: $C \mapsto A$

k_1	z_1 \ k_0 / z_0	0 C	3 A_-	4 E_+	7 G_+	8 E_-	9 A_+
0	A						
3	C_+		✖				✖
7	E_+, D_-				✖		
9	C_-		✖				✖

8. Major Ascending Third in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 4$

CF: $C \mapsto E$

k_1	z_1 \ k_0 / z_0	0 C	3 A_-	4 E_+	7 G_+	8 E_-	9 A_+
0	E			✖			
3	G_+						
4	C_-			✖			
7	A_-				✖		
8	C_+			✖			
9	G_-						

9. Major Descending Third in Cantus Firmus: $x_0 \mapsto x_1 = x_0 - 4$

CF: $E \mapsto C$

k_1	z_1 \ k_0 / z_0	0 E	3 G_+	4 C_-	7 A_-	8 C_+	9 G_-
0	C			✖			
3	A_-						
4	E_+			✖			
7	G_+				✖		
8	E_-			✖			
9	A_+						

10. Ascending Fourth in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 5$

CF: $D \mapsto G$

k_1	z_1 \ k_0 / z_0	0 D	7 A_+, G_-
0	G		
3	E_-		
7	D_+, C_-		✖
9	E_+		

CF: $E \mapsto A$

k_1	z_1 \ k_0 / z_0	0 E	3 G_+	4 C_-	7 A_-	8 C_+	9 G_-
0	A						
3	C_+						
7	E_+, D_-				✖		
9	C_-						

CF: $G \mapsto C$

k_1	z_1 \ k_0 / z_0	0 G	3 E_-	7 D_+, C_-	9 E_+
0	C				
3	A_-				
4	E_+				
7	G_+			✖	
8	E_-				
9	A_+				

CF: $A \mapsto D$

k_1	z_1 \ k_0 / z_0	0 A	3 C_+	7 E_+, D_-	9 C_-
0	D				
7	A_+, G_-			✖	

11. Descending Fourth in Cantus Firmus: $x_0 \mapsto x_1 = x_0 - 5$

CF: $D \mapsto A$

k_1	z_1 \ k_0 / z_0	0 D	7 A_+, G_-
0	A		
3	C_+		
7	E_+, D_-		✖
9	C_-		

CF: $G \mapsto D$

k_1	z_1 \ k_0 / z_0	0 G	3 E_-	7 D_+, C_-	9 E_+
0	D				
7	A_+, G_-			✖	

CF: $A \mapsto E$

k_1	z_1 \ k_0 / z_0	0 A	3 C_+	7 E_+, D_-	9 C_-
0	E				
3	G_+				
4	C_-				
7	A_-			✖	
8	C_+				
9	G_-				

CF: $C \mapsto G$

k_1	z_1 \ k_0 / z_0	0 C	3 A_-	4 E_+	7 G_+	8 E_-	9 A_+
0	G						
3	E_-						
7	D_+, C_-				✖		
9	E_+						

12. Ascending Tritone in Cantus Firmus: $x_0 \mapsto x_1 = x_0 + 6$

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